THE TEACHING OF MATHEMATICS
2024, Vol. XXVII, No. 1, pp. 27–32
DOI: 10.57016/TM-XDVI2817

A CHAIN OF EIGHT INEQUALITIES INVOLVING MEANS OF TWO ARGUMENTS

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Abstract. For two positive real numbers a and b, let $H := H(a,b)$, $G := G(a,b)$, $A := A(a,b)$ and $Q : Q(a,b)$ be the harmonic mean, the geometric mean, the arithmetic mean and the quadratic mean of a and b, respectively. In this short note, we prove the following interesting chain involving eight inequalities:

$$G \le \sqrt{QH} \le \sqrt{AG} \le \frac{A+G}{2} \le \frac{Q+H}{2} \le \sqrt{\frac{A^2+G^2}{2}} \le \sqrt{\frac{Q^2+H^2}{2}} \le \frac{Q+G}{2} \le A,$$

where equality holds in each of these inequalities if and only if $a = b$. Some remarks, in particular connected with Muirhead's inequality, and two questions related to a similar form of chain of inequalities are also given.

MathEduc Subject Classification: H34

AMS Subject Classification: 97H30

Key words and phrases: Harmonic mean; geometric mean, arithmetic mean; quadratic mean; H-G-A-Q inequality; Muirhead's inequality.

1. The result and its prof

For two positive real numbers a and b, let $H := H(a,b) = 2ab/(a+b)$, $G := G(a,b) = \sqrt{ab}$, $A := A(a,b) = (a+b)/2$ and $Q := Q(a,b) = \sqrt{(a^2+b^2)/2}$ be the harmonic mean, the geometric mean, the arithmetic mean and the quadratic mean of a and b, respectively. Then by the well-known harmonic mean-geometric mean-arithmetic mean-quadratic mean inequality ($H - G - A - Q$ inequality),

$$H \le G \le A \le Q, \tag{1}$$

with equality if and only if $a = b$.

In this note, we prove the following result involving a chain of eight inequalities.

THEOREM. *Under above notation, the following inequalities hold:*

$$G \le \sqrt{QH} \le \sqrt{AG} \le \frac{A+G}{2} \le \frac{Q+H}{2} \le \sqrt{\frac{A^2+G^2}{2}} \le \sqrt{\frac{Q^2+H^2}{2}} \le \frac{Q+G}{2} \le A, \tag{2}$$

where equality holds in each of these inequalities if and only if $a = b$.

Proof. Applying the $H-G-A-Q$ inequality given by (1), we find that

$$G=\sqrt{AH}\le\sqrt{QH},\quad \sqrt{AG}\le\frac{A+G}{2}\quad\text{and}\quad \frac{Q+G}{2}\le\sqrt{\frac{Q^2+G^2}{2}}=A. \tag{3}$$

Further, using the inequalities (1), the binomial expansions

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

and

$$(a-b)^8=a^8-8a^7b+28a^6b^2-56a^5b^3+70a^4b^4-56a^3b^5+28a^2b^6-8ab^7+b^8,$$

after calculations and factorizations by `Mathematica 11`, we find that

$$\sqrt{QH}\le\sqrt{AG}\iff A^2G^2-Q^2H^2\ge 0\iff ab(a-b)^4/4(a+b)^2\ge 0,$$

$$\begin{aligned}&(A+G)/2\le(Q+H)/2\iff(Q-G)^2\ge(A-H)^2\\ &\iff(Q^2+G^2-A^2+2AH-H^2)^2\ge 4Q^2G^2\iff(a-b)^8/16(a+b)^4\ge 0,\end{aligned}$$

$$\begin{aligned}&(Q+H)/2\le\sqrt{(A^2+G^2)/2}\iff 2A^2+2G^2\ge Q^2+2QH+H^2\\ &\iff(2A^2+2G^2-Q^2-H^2)^2\ge 4Q^2H^2\iff a^2b^2(a-b)^4/(a+b)^4\ge 0,\end{aligned} \tag{4}$$

$$\begin{aligned}&\sqrt{(A^2+G^2)/2}\le\sqrt{(Q^2+H^2)/2}\iff Q^2+H^2-G^2-A^2\ge 0\\ &\iff(a-b)^4/4(a+b)^2\ge 0,\end{aligned}$$

$$\begin{aligned}&\sqrt{(Q^2+H^2)/2}\le(Q+G)/2\iff 4Q^2G^2-(2A^2+G^2-Q^2)^2\ge 0\\ &\iff 2ab(a-b)^2\ge 0.\end{aligned}$$

The inequalities (3) and (4) immediately imply the chain of inequalities given by (2), and obviously, equality holds in any of these inequalities if and only if $a=b$. ■

2. Muirhead's inequality and its applications

Muirhead's inequality is an important generalization of the arithmetic-geometric mean inequality for n positive integers ($n=2,3,\dots$). It is a powerful tool for solving numerous inequality problems. In order to give Muirherad's inequality, we will need some definitions and related notations given as follows.

Let $x_1,x_2,\dots,x_n$ be positive real numbers and let $p=(p_1,p_2,\dots,p_n)\in\mathbb{R}^n$, where n is a positive integer. Then the *p-mean* of $x_1,x_2,\dots,x_n$ is defined by

$$[p]:=\frac{1}{n!}\sum_{\sigma\in S_n}x_{\sigma(1)}^{p_1}x_{\sigma(2)}^{p_2}\cdots x_{\sigma(n)}^{p_n},$$

where S_n is the set of all permutations of the set $\{1, 2, \ldots, n\}$ and the summation ranges over all $n!$ permutations $\sigma \in S_n$.

For example, $[(1, 0, \ldots, 0)] = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the arithmetic mean of $x_1, x_2, \ldots, x_n$ and $[(1/n, 1/n, \ldots, 1/n)] = x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n}$ is their geometric mean.

The above sum involved in the expression for $[p]$ is in Combinatorics often written as

$$\sum_{sym} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}.$$

See [2, Definition 2], where this sum is denoted as $T[p_1, p_2, \ldots, p_n](x_1, x_2, \ldots, x_n)$. For example, if $p = (3, 1, 0)$, then

$$\begin{aligned}\sum_{sym} x^3y^1z^0 &= x^3y^1z^0 + x^3z^1y^0 + y^3x^1z^0 + y^3z^1x^0 + z^3x^1y^0 + z^3y^1x^0\\ &= x^3y + x^3z + y^3x + y^3z + z^3x + z^3y.\end{aligned}$$

Next we introduce the concept of majorization in $\mathbb{R}^n$. Suppose that the vectors $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ and $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ satisfy the following conditions:

1) $p_1 \geq p_2 \geq \cdots \geq p_n$ and $q_1 \geq q_2 \geq \cdots \geq q_n$;

2) $p_1 \geq q_1$, $p_1 + p_2 \geq q_1 + q_2$, $\ldots$, $p_1 + p_2 + \cdots + p_{n-1} \geq q_1 + q_2 + \cdots + q_{n-1}$ and

3) $p_1 + p_2 + \cdots + p_n = q_1 + q_2 + \cdots + q_n$.

Then we say that $(p_1, p_2, \ldots, p_n)$ majorizes $(q_1, q_2, \ldots, q_n)$ and write $(p_1, p_2, \ldots, p_n) \succ (q_1, q_2, \ldots, q_n)$ (or $(q_1, q_2, \ldots, q_n) \prec (p_1, p_2, \ldots, p_n)$).

Under above notations and notions, *Muirhead's inequality* ([3]; also see [1] and [2, Theorem 4]) states that if $x_1, x_2, \ldots, x_n$ are positive real numbers and $p, q \in \mathbb{R}^n$ are such that $p \succ q$, then $[p] \geq [q]$. Furthermore, if $p \neq q$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Since $(1, 0, \ldots, 0) \succ (1/n, 1/n, \ldots, 1/n)$, it follows that the arithmetic-geometric mean inequality is a consequence of Muirhead's inequality.

Example. Let x and y be positive real numbers. Since $(2, 2) \prec (3, 1)$, by Muirhead's inequality (in the sequel, shortly denoted by MI), we have $\sum_{sym} x^2y^2 \leq \sum_{sym} x^3y^1$, i.e., $2x^2y^2 \leq x^3y + xy^3$, or equivalently, $4x^2y^2 \leq xy(x + y)^2$, which implies the $H-G$ inequality $2xy/(x+y) \leq \sqrt{xy}$. As it is noticed above (for arbitrary n instead of 2), since $(1/2, 1/2) \prec (1, 0)$, MI implies the $G - A$ inequality $\sqrt{xy} \leq (x + y)/2$. Similarly, because of $(1, 1) \prec (2, 0)$, using MI we obtain $\sum_{sym} x^1y^1 \leq \sum_{sym} x^2y^0$, i.e., $2xy \leq x^2 + y^2$, which is equivalent to $(x + y)^2 \leq 2(x^2 + y^2)$, which implies the $A-Q$ inequality $(x+y)/2 \leq \sqrt{(x^2 + y^2)/2}$. Hence, applying MI inequality, we have proven the $H - G - A - Q$ inequality for two positive integers.

Remark 1. A direct calculation, without using software `Mathematica 11`, shows that the last inequality from (4) of Theorem is equivalent to $a^2 + b^2 \geq 2ab$, which is, as it is showed in Example, an immediate consequence of MI.

Furthermore, the first, third and fourth inequality from (4) reduce to the following one:

$$a^4 + 6a^2b^2 + b^4 \geq 4a^3b + 4ab^3. \tag{5}$$

Since $(4,0) \succ (3,1)$, by MI we obtain $2(a^4 + b^4) \geq 2(a^3b + ab^3)$, i.e., $a^4 + b^4 \geq a^3b + ab^3$. On the other hand, since $(3,1) \succ (2,2)$, by MI we get $a^3b + b^3a \geq 2a^2b^2$, so $6a^2b^2 \leq 3a^3b + 3b^3a$. Since the previous two inequalities obviously do not imply the inequality (5), it follows that the first, third and fourth inequality of (4) cannot be derived applying MI.

Finally, the second inequality from (4) reduces to the following one:

$$a^8 + 28a^6b^2 + 70a^4b^4 + 28a^2b^6 + b^8 \geq 8a^7b + 56a^5b^3 + 56a^3b^5 + 8ab^7. \tag{6}$$

Since $(8,0) \succ (7,1)$, by MI we obtain $2(a^8 + b^8) \geq 2(a^7b + ab^7)$, i.e., $a^8 + b^8 \geq a^7b + ab^7$. On the other hand, since $(6,2) \succ (5,3)$, by MI we find that $2(a^6b^2 + a^2b^6) \geq 2(a^5b^3 + a^3b^5)$, whence it follows that $28a^6b^2 + 28a^2b^6 \geq 28a^5b^3 + 28a^3b^5$. Adding the previous two inequalities, we obtain

$$a^8 + 28a^6b^2 + 28a^2b^6 + b^8 \geq a^7b + 28a^5b^3 + 28a^3b^5 + ab^7. \tag{7}$$

Furthermore, since $(7,1) \succ (4,4)$, by MI we find that $a^7b + ab^7 \geq 2a^4b^4$, whence it follows that $7a^7b + 7ab^7 \geq 14a^4b^4$. As $(5,3) \succ (4,4)$, by MI we obtain $a^5b^3 + a^3b^5 \geq 2a^4b^4$, whence it follows that $28a^5b^3 + 28a^3b^5 \geq 56a^4b^4$. Adding the previous two inequalities, we obtain

$$70a^4b^4 \leq 7a^7b + 27a^5b^3 + 28a^3b^5 + 7ab^7. \tag{8}$$

Obviously, the inequalities (7) and (8) do not imply the inequality (6). Clearly, this would be true if the converse of the inequality (8) were true. Hence, the second inequality of (4) cannot be derived applying MI.

By the binomial expansion, the obvious inequality $(a-b)^{2n} \geq 0 \ \ (n = 1, 2, \ldots)$ is equivalent to the following one:

$$\sum_{k=0}^{n} \binom{2n}{2k} a^{2n-2k} b^{2k} \geq \sum_{k=1}^{n} \binom{2n}{2k-1} a^{2n-2k+1} b^{2k-1}. \tag{9}$$

If $n \geq 2$ is an even integer, $n = 2m \ \ (m = 1, 2, \ldots)$, then the central term of the binomial expansion $(a-b)^{4m}$ is $\binom{4m}{2m} a^{2m} b^{2m}$ and this term belongs to the sum on the left-hand side of (9). Note that $2m < \max_{1 \leq k \leq 2m} \{4m - 2k + 1, 2k - 1\}$. Namely, without loss of generality, we can suppose that $2k - 1 \geq 4m - 2k + 1$, i.e., $2m \leq 2k - 1$, and since $2m$ is even, it must be $2m \leq 2k - 2$. It follows that $(4m - 2k + 1, 2k - 1) \succ (2m, 2m)$ for all $k = 1, \ldots, m$. This shows that if $a \neq b$, MI cannot be applied to prove that $\binom{4m}{2m} a^{2m} b^{2m}$ is greater or equal to a sum of some terms on the right-hand side of the inequality (9).

If $n > 2$ is an odd integer, $n = 2m + 1 \ \ (m = 1, 2, \ldots)$, then the sum of the first term and the last term on the right-hand side of the inequality (9) is equal to $(4m+2)(a^{4m+1}b+ab^{4m+1})$. The sum of the first term and the last term on the left-hand side of (9) is equal to $a^{4m+2}+b^{4m+2}$ and by Muirhead's inequality, $a^{4m+2}+b^{4m+2} \geq a^{4m+1}b+ab^{4m+1}$. Hence, in order to prove the inequality (9) applying MI, it would be necessary to show that $(4m+1)(a^{4m+1}b+ab^{4m+1})$ is less or equal to a sum of some terms involved in the left-hand side of (9). However, this is impossible because $4m+1 > \max_{1\leq k\leq 2m}\{4m+2-2k, 2k\}$ implies that $(4m+1,1) \succ (4m+2-2k, 2k)$ for all $k=1,\dots,m$.

Therefore, the inequality (9) can be proved applying Muirhead's inequality only for $n=1$.

3. Concluding remarks and questions

Remark 2. Notice that using the inequalities (1), we obviously have $H \leq \sqrt{GH} \leq (G+H)/2 \leq \sqrt{(G^2+H^2)/2} \leq G$, and hence, the chain of inequalities given by (2) can be extended from the left-hand side with these four inequalities. Similarly, by the inequalities (1), we obviously have $A \leq \sqrt{QA} \leq (Q+A)/2 \leq \sqrt{(Q^2+A^2)/2} \leq Q$, and thus, the chain of inequalities given by (2) can be extended from the right-hand side with these four inequalities. Therefore, the chain of inequalities given by (2) can be extended to the chain of inequalities involving sixteen inequalities (see (10)).

Remark 3. Let x and y be positive real variables, and let $M := M(x,y)$ be an arbitrary mean of x and y, where $M \in \{H,G,A,Q\}$. Furthermore, if a and b are positive real numbers and if $M_1, M_2, M_3 \in \{H,G,A,Q\}$, where two or all three means M_k $(k=1,2,3)$ may coincide, then by convention we define $M_1(M_2,M_3) := M_1(M_2(a,b), M_3(a,b))$. For example,

$$A(G,Q) = (\sqrt{ab}+\sqrt{(a^2+b^2)/2})/2, \quad H(A,G) = 2(a+b)\sqrt{ab}(a+b+2\sqrt{ab}),$$
$$Q(G,A) = \sqrt{(a^2+b^2+6ab)/8}, \;\; G(G,A) = \sqrt[4]{ab(a+b)^2/4}, \;\; G(H,A) = \sqrt{ab} = G.$$

Clearly, it holds $M_1(M_2,M_3) = M_1(M_3,M_2)$ for all $M_1,M_2,M_3 \in \{H,G,A,Q\}$. Notice that $M_1(M_2,M_2) = M_2$ for all $M_1,M_2 \in \{H,G,A,Q\}$. An easy verification shows that in the set

$$\{\, M_1(M_2,M_3) : M_1,M_2,M_3 \in \{H,G,A,Q\} \text{ and } M_2 > M_3 \,\}$$

there are only two elements which coincide to some mean from the set $\{H,G,A,Q\}$: namely, $G(A,H) = G$ and $Q(Q,G) = A$. In accordance with the all previously noted, the number of different expressions of the form $M_1(M_2,M_3)$ with $M_1,M_2,M_3 \in \{H,G,A,Q\}$ such that $M_2 \geq M_3$ is equal to $4\cdot(3\cdot 2)+4-2 = 26$.

Note that in view of Remark 2, the chain of inequalities (2) from Theorem which contains eight inequalities can be extended to the chain of sixteen inequalities involving means of means of two arguments. Using the previous notations, this chain of inequalities can be written as follows.

$$\begin{aligned} H &\leq G(G,H) \leq A(G,H) \leq Q(G,H) \leq G \leq G(Q,H) \\ &\leq G(A,G) \leq A(A,G) \leq A(Q,H) \leq Q(A,G) \leq Q(Q,H) \\ &\leq A(Q,G) \leq A \leq G(Q,A) \leq A(Q,A) \leq Q(Q,A) \leq Q. \end{aligned} \tag{10}$$

We say that the inequality $M_1(M_2, M_3) \leq M_4(M_5, M_6)$ such that $M_k \in \{H, G, A, Q\}$ $(k = 1, \dots, 6)$, $M_2 \geq M_3$ and $M_5 \geq M_6$, is trivial if $M_1 \leq M_4$, $M_2 \leq M_5$ and $M_3 \leq M_6$. Clearly, each trivial inequality follows immediately from the inequality (1). Notice that the first four, as well as the last four inequalities involved in (10), and the inequality $G(A, G) \leq A(A, G)$ from (2) and (10) are trivial. Thus, both chains of inequalities (2) and (10) can be reduced to the chain involving seven non-trivial inequalities.

Finally, we propose the following two curious questions concerning the chains of inequalities whose terms belong to the set $\mathcal{M} := \{\, M_1(M_2, M_3) : M_1, M_2, M_3 \in \{H, G, A, Q\}\,\}$ whose total number is 26.

QUESTION 1. Is there a chain of inequalities that contains more than sixteen inequalities whose members are from the set $\mathcal{M}$?

QUESTION 2. Is there a chain of inequalities that contains more than seven non-trivial inequalities whose members are from the set $\mathcal{M}$?

ACKNOWLEDGEMENTS. The author is thankful to the referee for useful suggestions.

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Received: 14.01.2024, in revised form 17.01.2024

Accepted: 27.03.2024