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FINITE GENERATIVITY OF HOMOLOGY AND COHOMOLOGY MODULES

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Abstract. In this paper, we consider the following question: if all homology groups of a space X are finitely generated, and if R is a commutative ring with identity, is it true that the homology and cohomology R-modules $H_i(X;R)$ and $H^i(X;R)$ are also finitely generated? We show that the answer to this question is negative in general, but affirmative if R is an integral domain. In the case when R is a principal ideal domain, and $H_i(X;R)$ is finitely generated for all i, we also discuss computing $H_i(X;M)$ and $H^i(X;M)$ for a finitely generated R-module M.

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1. Introduction

Basic homology techniques have been a standard part of the mathematics curriculum at universities for several decades. They are usually presented within introductory courses of topology through the study of homology and cohomology groups of topological spaces. This paper deals with homology and cohomology groups with coefficients in a ring, and more generally, in a module over a ring. In particular, we aim to provide the reader with a deeper insight into homology techniques through examining finite generativity of resulting homology and cohomology modules.

A topological space X is said to be of *finite type* if all its homology groups $H_i(X)$, $i \geqslant 0$, are finitely generated. For such an X, the universal coefficient theorems (for homology and cohomology) provide an algorithm for computing $H_i(X;G)$ and $H^i(X;G)$, at least when the abelian group G is itself finitely generated. If X is of finite type and G finitely generated, it is a consequence of this algorithm that $H_i(X;G)$ and $H^i(X;G)$ are finitely generated abelian groups.

If we take the coefficients in a commutative ring (with identity) R, then the groups $H_i(X;R)$ and $H^i(X;R)$ have the additional structure of R-modules, which in return gives us more information about the topological space X itself. A natural question that arises in teaching algebraic topology and homological algebra is whether these R-modules are necessarily finitely generated. In considering this question, we illustrate an interesting interplay between the notions of abelian group and R-module which is instructive for curious students who would like to know more about the theory of homology and cohomology.

It is straightforward from the algorithm (i.e. from the universal coefficient theorems) that the answer to the proposed question is positive if the ring R has no zero divisors. We prove this fact in Theorem 2.1. However, if R does have zero divisors, then the answer is negative. We provide an example of such a ring R by a slight modification of one of the first examples of non-Noetherian rings—namely, the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ over infinitely many variables—while for the space X we take an appropriate Moore space. Moreover, in Theorem 2.3 we present a whole array of examples.

If R is a PID (principal ideal domain), then the universal coefficient theorems hold in the category of R-modules as well (see [3, p. 222, 243]). So, if we start with homology modules $H_i(X;R)$ and if we take an R-module M, then these theorems express $H_i(X;M)$ and $H^i(X;M)$ in terms of $H_i(X;R)$ and M, via tensor and torsion products, and functors Hom_R and Ext_R. Now, if M and all $H_i(X;R)$, $i \geqslant 0$, are finitely generated, then they are finite direct sums of cyclic modules, and in order to obtain an algorithm for computing $H_i(X;M)$ and $H^i(X;M)$ (analogous to the one for abelian groups), we need to know how to calculate tensor and torsion products, as well as Hom_R and Ext_R, of two cyclic R-modules. A key ingredient in that regard is Proposition 3.1. This proposition generalizes the well-known fact for cyclic groups that, if m and n are positive integers and $\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} n\mathbb{Z}/m\mathbb{Z}$ the multiplication with n, then both kernel and cokernel of this map are isomorphic to the cyclic group $\mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of m and n.

2. Are $H_i(X;R)$ and $H^i(X;R)$ finitely generated R-modules?

The universal coefficient theorem for homology [2, Theorem 3A.3] states that if X is a space and G an abelian group, then for every i there is a split short exact sequence of abelian groups

$$0 \to H_i(X) \otimes G \to H_i(X;G) \to \mathrm{Tor}(H_{i-1}(X), G) \to 0.$$

Therefore, for all i we have an isomorphism:

$$H_i(X;G) \cong (H_i(X) \otimes G) \oplus \mathrm{Tor}(H_{i-1}(X), G). \tag{2.1}$$

Similarly, the cohomology variant of the theorem [2, Theorem 3.2] establishes the split short exact sequence

$$0 \to \mathrm{Ext}(H_{i-1}(X), G) \to H^i(X;G) \to \mathrm{Hom}(H_i(X), G) \to 0,$$

and consequently, an isomorphism

$$H^i(X;G) \cong \mathrm{Ext}(H_{i-1}(X), G) \oplus \mathrm{Hom}(H_i(X), G). \tag{2.2}$$

Also, it is well known that all the operations on abelian groups appearing in (2.1) and (2.2) behave nicely with respect to finite direct sums (see [2, p. 192, 195, 215, 265]). So, if the space X is of finite type, then every group $H_i(X)$ is isomorphic to a finite direct sum of cyclic groups, and in order to determine $H_i(X;G)$ and $H^i(X;G)$, it suffices to determine $C \otimes G$, $\mathrm{Tor}(C,G)$, $\mathrm{Ext}(C,G)$ and $\mathrm{Hom}(C,G)$, where C is a cyclic group. The following isomorphisms can be found in [2, p. 195, 215, 265]:

$$\begin{aligned} &\mathbb{Z} \otimes G \cong G, \qquad \mathrm{Ext}(\mathbb{Z}, G) = 0, \\ &\mathbb{Z}/n\mathbb{Z} \otimes G \cong \mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong \mathrm{coker}\left(G \xrightarrow{n} G\right); \end{aligned} \tag{2.3}$$

$$\text{(2.4)}\qquad \begin{aligned} &\operatorname{Hom}(\mathbb{Z},G)\cong G, \qquad \operatorname{Tor}(\mathbb{Z},G)=0,\\ &\operatorname{Hom}(\mathbb{Z}/n\mathbb{Z},G)\cong \operatorname{Tor}(\mathbb{Z}/n\mathbb{Z},G)\cong \ker\big(G\xrightarrow{n}G\big); \end{aligned}$$

where the map $G \xrightarrow{n} G$ is the multiplication with the positive integer n.

Now, let R be a commutative ring with identity. Since R is a module over itself, for every abelian group A, each of the abelian groups $A\otimes R$, $\operatorname{Tor}(A,R)$, $\operatorname{Hom}(A,R)$ and $\operatorname{Ext}(A,R)$ has a natural structure of an R-module. Additionally, for any space X the abelian groups $H_i(X;R)$ and $H^i(X;R)$, $i \geqslant 0$, are also R-modules. Moreover, if G is replaced by R in the relations mentioned above, all the relevant isomorphisms are R-module isomorphisms.

We want to answer the question whether the R-modules $H_i(X;R)$ and $H^i(X;R)$ must be finitely generated if X is of finite type. It follows from the previous discussion that this amounts to checking whether $\ker\big(R \xrightarrow{n} R\big)$ and $\operatorname{coker}\big(R \xrightarrow{n} R\big)$ are finitely generated R-modules.

Since a quotient of a finitely generated module is finitely generated (the cosets of generators for the module generate the quotient), $\operatorname{coker}\big(R \xrightarrow{n} R\big) = R/nR$ is a finitely generated R-module for each n (moreover, the R-module R/nR is cyclic—it is generated by a single element, namely $1+nR$). On the other hand, in general, the situation for $\ker\big(R \xrightarrow{n} R\big)$ is more complicated. However, if R is an integral domain, i.e., if it has no zero divisors, the R-module $\ker\big(R \xrightarrow{n} R\big)$ is finitely generated, which is the crux of the proof of the following theorem.

THEOREM 2.1. *If R is an integral domain and X a space of finite type, then $H_i(X;R)$ and $H^i(X;R)$ are finitely generated R-modules.*

Proof. As we have stated above, we have the isomorphism of R-modules

$$H_i(X;R) \cong (H_i(X)\otimes R) \oplus \operatorname{Tor}(H_{i-1}(X),R),$$

and since $H_i(X)$ and $H_{i-1}(X)$ are finitely generated, according to previous discussion $H_i(X;R)$ is a finite direct sum of R-modules of form

$$\mathbb{Z}\otimes R\cong R,\quad \mathbb{Z}/n\mathbb{Z}\otimes R \cong \operatorname{coker}\big(R\xrightarrow{n}R\big),\quad \operatorname{Tor}(\mathbb{Z},R)=0\quad\text{and}$$
$$\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z},R)\cong \ker\big(R\xrightarrow{n}R\big) = \begin{cases} R, & n=0 \text{ in } R\\ 0, & n\neq 0 \text{ in } R\end{cases}$$

and all of these are cyclic R-modules. Note that in obtaining the last equality, we have used the assumption that R has no zero divisors.

Similarly, for cohomology we have the isomorphism of R-modules

$$H^i(X;R)\cong \operatorname{Hom}(H_i(X),R)\oplus \operatorname{Ext}(H_{i-1}(X),R),$$

so $H^i(X;R)$ is a finite direct sum of cyclic R-modules as well. ■

However, the R-module $\ker\big(R \xrightarrow{n} R\big)$ need not be finitely generated in general. Let us construct an example to demonstrate this. Since $\ker\big(R \xrightarrow{n} R\big)$ is an ideal in R, that is, a submodule of the R-module R, first we need a non-Noetherian ring R.

One of the first such rings that comes to mind is the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$— the ideal $(x_1, x_2, \dots)$ generated by x_1, x_2, ... (which consists of the polynomials with vanishing constant term) is not finitely generated, but $\mathbb{Z}[x_1, x_2, \dots]$ is an integral domain, hence by Theorem 2.1 it cannot assume the role of the ring R in the example we are looking for.

However, an appropriate quotient of $\mathbb{Z}[x_1, x_2, \dots]$ fits our needs.

LEMMA 2.2. *Let $n \geqslant 2$ be an integer. If R is the quotient of the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ by the ideal generated by $nx_1, nx_2, \dots$, i.e.*

$$R = \mathbb{Z}[x_1, x_2, \dots]/(nx_1, nx_2, \dots),$$

then the ideal $\ker\big(R \xrightarrow{n} R\big)$ *in R is not finitely generated.*

Proof. We can think of each element of R as a polynomial whose constant term is an integer, while all other coefficients are from the ring $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n. Then the ideal $\ker\big(R \xrightarrow{n} R\big)$ consists of the polynomials with zero constant term, and so, it is the ideal generated by (the cosets of) the variables $x_1, x_2, \dots$

Let us denote this ideal by I, and show that it is not finitely generated. Suppose to the contrary that some polynomials $f_1, f_2, \dots, f_k \in I$ generate I. Every polynomial in infinitely many variables is a *finite* sum of monomials, and therefore includes only finitely many of the variables. Let x_m be a variable which does not appear in neither of the polynomials $f_1, f_2, \dots, f_k$. Since $x_m \in I$, we have

$$x_m = \sum_{j=1}^{k} p_j f_j, \tag{2.5}$$

for some $p_1, p_2, \dots, p_k \in R$. So there exists $j \in \{1, 2, \dots, k\}$, a monomial in p_j and a monomial in f_j whose product is a constant times x_m. However, every monomial in f_j has positive degree (since $f_j \in I$) and does not contain the variable x_m. Therefore, the equation (2.5) is impossible, and the lemma is proved. ∎

Recall that for a positive integer m and an abelian group G there is a space X with the property

$$H_i(X) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ G, & i = m \\ 0, & \text{otherwise} \end{cases}$$

Such a space is called a Moore space of type (G, m), and is denoted by $M(G, m)$ (see [2, p. 143]. Obviously, $M(G, m)$ is of finite type if and only if G is finitely generated.

THEOREM 2.3. *Let $n \geqslant 2$ be an integer and let $R = \mathbb{Z}[x_1, x_2, \dots]/(nx_1, nx_2, \dots)$ be the ring from Lemma 2.2. Then, for all positive integers m, the R-modules*

$$H_{m+1}\big(M(\mathbb{Z}/n\mathbb{Z}, m); R\big) \quad \text{and} \quad H^m\big(M(\mathbb{Z}/n\mathbb{Z}, m); R\big)$$

are not finitely generated.

Proof. By the universal coefficient theorems (2.1) and (2.2), as well as the properties (2.3) and (2.4), both of these R-modules are isomorphic to $\ker\big(R \xrightarrow{n} R\big)$. The result follows from Lemma 2.2. ∎

3. The PID case

All the isomorphisms featuring in (2.1)–(2.4) are isomorphisms in the category of abelian groups, which is the same as the category of modules over the ring of integers $\mathbb{Z}$. As we have already stated, every finitely generated abelian group is isomorphic to a finite direct sum of cyclic ones. On the other hand, the groups $\ker\big(G \xrightarrow{n} G\big)$ and $\operatorname{coker}\big(G \xrightarrow{n} G\big)$ (from (2.3) and (2.4) are easily calculated if G is cyclic. Namely, $\ker\big(\mathbb{Z} \xrightarrow{n} \mathbb{Z}\big) = 0$, $\operatorname{coker}\big(\mathbb{Z} \xrightarrow{n} \mathbb{Z}\big) = \mathbb{Z}/n\mathbb{Z}$, while

$$\ker\big(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}\big) \cong \operatorname{coker}\big(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}\big) \cong \mathbb{Z}/d\mathbb{Z}, \tag{3.1}$$

where d is the greatest common divisor of positive integers m and n. Therefore, we have an algorithm for computing the groups $H_i(X;G)$ and $H^i(X;G)$ in the case when X is of finite type and G finitely generated.

In this section we show that the same algorithm works more generally—in the category of modules over a principal ideal domain (PID).

Let R be a PID. The universal coefficient theorems (2.1) and (2.2) hold in this generalized setting as well. Namely, for a space X and an R-module M, if we substitute $H_i(X;R)$ for $H_i(X)$ and M for G in (2.1) and (2.2), we obtain R-module isomorphisms (see [3, p. 222, 243]):

$$H_i(X;M) \cong (H_i(X;R) \otimes_R M) \oplus \operatorname{Tor}_R(H_{i-1}(X;R), M), \tag{3.2}$$

$$H^i(X;M) \cong \operatorname{Ext}_R(H_{i-1}(X;R), M) \oplus \operatorname{Hom}_R(H_i(X;R), M). \tag{3.3}$$

Now let X be a space such that homology modules $H_i(X;R)$ are finitely generated (by Theorem 2.1, this is the case if X is of finite type). If M is finitely generated as well, we want to present an algorithm for computing $H_i(X;M)$ and $H^i(X;M)$ that generalizes the one above for abelian groups ($\mathbb{Z}$-modules). Recall that every finitely generated module over a PID is isomorphic to a (finite) direct sum of cyclic modules (see [1, Theorem 5, p. 462]). On the other hand, the operations $\otimes_R$, Tor_R, Hom_R and Ext_R commute with finite direct sums (see [3]), and so as in the case of abelian groups, it suffices to determine $C_1 \otimes_R C_2$, $\operatorname{Tor}_R(C_1, C_2)$, $\operatorname{Hom}_R(C_1, C_2)$ and $\operatorname{Ext}_R(C_1, C_2)$ for two cyclic modules C_1 and C_2.

If C is a cyclic R-module, then there exists an epimorphism $\varphi : R \to C$ from the free cyclic R-module R onto C. By the first isomorphism theorem, $C \cong R/\ker\varphi$, and since R is a PID, the ideal $\ker\varphi$ is principal, that is, $\ker\varphi = aR$ for some $a \in R$. We conclude that every cyclic R-module is isomorphic to R/aR for some $a \in R$ (the case $a = 0$ corresponds to a *free* cyclic module).

The isomorphisms analogous to those from (2.3) and (2.4) hold in the category of R-modules (see [3, p. 221, 241, 242]):

$$R \otimes_R M \cong M, \qquad \operatorname{Ext}_R(R, M) = 0,$$

$$R/aR \otimes_R M \cong \operatorname{Ext}_R(R/aR, M) \cong \operatorname{coker}\big(M \xrightarrow{a} M\big);$$

$$\operatorname{Hom}_R(R, M) \cong M, \qquad \operatorname{Tor}_R(R, M) = 0,$$

$$\operatorname{Hom}_R(R/aR, M) \cong \operatorname{Tor}_R(R/aR, M) \cong \ker\big(M \xrightarrow{a} M\big),$$

where the map $M \xrightarrow{a} M$ is the multiplication with the nonzero element $a \in R$.

By the previous discussion, it is enough to calculate these modules in the case when M is cyclic, i.e. isomorphic to either R or R/bR for some nonzero $b \in R$. Therefore, we are left to determine kernels and cokernels of the maps $R \xrightarrow{a} R$ and $R/bR \xrightarrow{a} R/bR$. The kernel of the former map is trivial since R has no zero divisors, and its cokernel is obviously the cyclic module R/aR. The following proposition calculates the kernel and the cokernel of the latter map (generalizing (3.1)), and thus completes the announced algorithm.

Proposition 3.1. *If R is a PID, and $a, b \in R \setminus \{0\}$, then we have the following isomorphisms of R-modules:*

$$\ker\left(R/bR \xrightarrow{a} R/bR\right) \cong \operatorname{coker}\left(R/bR \xrightarrow{a} R/bR\right) \cong R/dR,$$

where $d \in R$ is a greatest common divisor of a and b.

Proof. Let us firstly notice that

$$a(R/bR) = (aR + bR)\,/bR = dR/bR. \tag{3.4}$$

Indeed, $aR + bR = dR$ since R is a PID, and the first equality follows from the following equivalences (where $x \in R$):

$$\begin{aligned} x + bR \in a\,(R/bR) &\iff (\exists y \in R)\, x + bR = a(y + bR) = ay + bR \\ &\iff (\exists y \in R)\, x - ay \in bR \\ &\iff x \in aR + bR. \end{aligned}$$

Now it is straightforward from (3.4) that

$$\operatorname{coker}\left(R/bR \xrightarrow{a} R/bR\right) = (R/bR)\ /\ a(R/bR) = (R/bR)\ /\ (dR/bR) \cong R/dR, \tag{3.5}$$

by the third isomorphism theorem.

Now on to proving that the kernel of $R/bR \xrightarrow{a} R/bR$ is also isomorphic to R/dR. Let $s, t \in R$ be elements such that $a = sd$ and $b = td$. Since every PID is also a UFD (unique factorization domain), we have $\gcd(s, t) = 1$. Hence,

$$\begin{aligned} \ker\left(R/bR \xrightarrow{a} R/bR\right) &= \left\{x + bR \in R/bR \mid ax \in bR\right\} \\ &= \left\{x + bR \in R/bR \mid (\exists y \in R)\ \ ax = by\right\} \\ &= \left\{x + bR \in R/bR \mid (\exists y \in R)\ \ sdx = tdy\right\} \\ &= \left\{x + bR \in R/bR \mid\ t \,|\, sx\right\} = \left\{x + bR \in R/bR \mid\ t \,|\, x\right\} \\ &= tR/bR = tR/tdR \cong R/dR. \end{aligned} \tag{3.6}$$

The last isomorphism is a consequence of the fact that R has no zero divisors and that $t \neq 0$. Namely, this implies that the multiplication by t induces an isomorphism of R-modules R and tR, via which the submodule $dR \subseteq R$ corresponds to the submodule $tdR \subseteq tR$. ■

Therefore, we are able to compute $C_1 \otimes_R C_2$, $\mathrm{Tor}_R(C_1, C_2)$, $\mathrm{Hom}_R(C_1, C_2)$ and $\mathrm{Ext}_R(C_1, C_2)$ for any two cyclic modules C_1 and C_2. Note that all of these resulting modules are themselves cyclic. Therefore, we have the following corollary.

Corollary 3.2. *If R is a PID, X is a space such that homology modules $H_i(X; R)$, $i \geqslant 0$, are all finitely generated, and M is a finitely generated R-module, then the R-modules $H_i(X; M)$ and $H^i(X; M)$ are finitely generated as well.*

Remark 3.3. The results presented in this section are actually purely algebraic. Instead of a space with finitely generated homology R-modules, one can take a chain complex of R-modules with finitely generated homology, and consider its homology and cohomology with coefficients in a finitely generated R-modul M.

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