THE TEACHING OF MATHEMATICS
2000, Vol. III, 1, pp. 1–14

ON MATHEMATICAL FORMALISM IN HIGH SCHOOLS CURRICULA

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Abstract. Reflections of A. Ya. Khinchin (1894–1959) on: mathematical formalism; the need for proper connection between the content of mathematical facts and their outward, superficial descriptions; the three levels of mathematical knowledge and connections between them.

AMS Subject Classification: 00 A 35
Key words and phrases: Mathematical formalism, source, object of research, tools.

All existing studies about the quality of mathematical preparation of students graduating from high schools point out that the most frequent and the most glaring deficiency of such a preparation until now, is the lack of both, the knowledge of mathematical *formalism* and mathematical skills of graduates. This deficiency up to its highest degree, impedes the achievement of all other goals in teaching of mathematics in high schools. Before anywhere else, this deficiency shows itself in the most severe form in a practical application of the acquired knowledge and mathematical skills. The fact that one who graduated from a high school and who has learned there only superficial and formalistic manipulation of mathematical methods without comprehending their content, will undoubtedly be unable to solve a real life problem due to his/her inability to see which of these methods are needed for solving such a problem. Such a person is unable, as we would say, to *mathematically formulate* a practical problem; up to a great extent, such a person will be unable to *solve* the problem due to his lack of skills to realistically comprehend needed formal operations. As a consequence, neither the interest in solving a considered practical problem, nor even the mathematical formulation of the problem, can guide him/her during the selection process of needed operations.

To a no lesser extent does the lack of knowledge of the mathematical formalism hamper the performance of high school graduates when they enter institutions of higher learning. The higher mathematics which they now encounter, does not by its very essence allow a pure superficial, and formalistic approach. Such a person who has previously learned to manipulate only superficially and to use formulae when dealing with mathematics, will find himself or herself when faced with a dialectically living world of variable quantities, unable to comprehend anything of this world. Such a person who is without the ability to connect the superficial,

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This article of an outstanding mathematician A. Ya. Khinchin, who had an important influence on teaching in the Soviet period, is not easily available in its original form (Survey of the Academy of Pedagogical Sciences of the Russian Federation, vol. 4, (1946), pp. 7–20). Professor N. Kh. Rozov suggested us to issue an English translation and professor Z. R. Pop-Stojanović has done the present one. Our thanks go to them both, equally. (Editors) formal apparatus with the mathematical content standing behind it, is unable to solve a mathematical problem.

Perhaps it is the most difficult to accept the consequence of this formalistic knowledge of mathematics showing a complete lifelessness and uselessness in forming the scientific view of the world which our students need to acquire, and which is the most important goal of our schools providing general education. It is hardly needed to prove that the knowledge and skills connected only with superficial forms of a subject under study, and which are cut off from its content, can in any measure influence the formation of ideas in minds of students, as well as their view of the world. At best, they are only capable to simulate their drills of purely formalistic cognitive abilities.

Therefore, such a formal knowledge of mathematical knowledge acquired by our students, de facto serves as an obstacle toward the broad educational goal of our schools. Hence, there are no, and there can not be, different opinions concerning the question of the necessity and timing of the struggle against this state of mathematical education in our schools. However, in order for this struggle to succeed, it has to be conducted not in a crude, but in a solid, scientific way. Also, it would be hardly inappropriate to blame for such a state of education, elementary school teachers and early written papers on methodology of teaching with their well-known platitudes. Before anything else, it is necessary to perform a deep scientific analysis of the situation which we want to change; by a thorough analysis we have to identify its deep roots and its immediate causes, and only then to start a scientific foundation of the most effective method needed in this struggle.

In this process of studies, the theoretical thought of experts of methodology ought to go hand in hand with observation and experiment. The ensuing enormous task in dealing with this problem, should not hinder us, if our goal is to change completely the existing state, not merely to patch up things in a crude way which will not guaranty a solid result. I believe in particular, that Department of Mathematics of the Institute for the Methods of Education of the Academy of Pedagogical Sciences of the Russian Federation, has all what is needed to make the solution of this problem in the forthcoming years its most important goal, and whose search for the solution of the problem will attract the cooperation of many Departments of our strongest Pedagogical Institutes.

However, one has to admit that until now, this problem has not been posed as a scientific problem. In this article I do not intend to gather some definite conclusions about any of questions raised here. I see my mission in a quite different light: to present these material about which I was thinking for some time; its parts are quite of a preliminary nature, and I hope they will provoke a lively discussion. If in the course of this discussion, we are lucky enough to formulate the framework of the approach for solving the problem, and in the process we attract the attention for its solution of wide pedagogical circles, that would be all I have been hoping for.

Now, I am moving on to the basic problem—to answer the question what represents the formalistic mathematical knowledge. Often, in order to reveal the essence of a complex problem, it is useful to carefully analyse it on a small number of clear examples. Hence, we start by listing several such examples.

1. A student will speedily and correctly answer the question "what is a logarithm," while after the ample time given to him, he is unable to find, without use of tables, $10^{\log 7}$.

2. A student will correctly draw the graph of the logarithmic function. With this graph in front of him, he is unable to answer the question, "what will happen to the logarithm of a number as this number is decreasing to zero?"

3. A student will solve easily a system of equations with unknowns x and y. However, he is unable to solve the same system of equations in which unknowns are denoted by l and k.

4. A student will correctly prove a theorem from geometry with a familiar arrangement of a drawn figure used in the theorem. However, he is unable to prove the same theorem when a different, but an equivalent drawing, is presented.

After a careful analysis of these and similar examples, we come to a conclusion that in all of them *a proper connection between the content of mathematical facts and their outward, superficial descriptions, has been broken in student's cognitive process. (Verbally, symbolically, or visually.)* This correct connection, which is necessary to understand the essence, requires that the basic goal of the study is the object itself, that is, its inside content, while its outward expressions (given verbally, symbolically, or visually), are only means or tools for facilitating the cognition of usually difficult essential facts. In all these above given examples, (as well as in many others which are similar), this proper connection has been radically distorted. An outward expression of a mathematical fact, does not accept its subordinated role which is assigned to it by its very essence, but becomes a self-sufficient factor, often dominating the internal content. In the first two given examples, we find that the student lacks the scientific knowledge of the basic facts involved there. A student who does not know to find $10^{\log 7}$, *de facto does not know* what is a logarithm, despite his fluent declamation of its definition. This definition is for our student an empty phrase which is in no way connected with the basic concept of the logarithm, (although for solving the posed problem, it is necessary only to *know* what is a logarithm). Similarly, in the second example, while drawing the graph of the logarithmic function correctly, our student de facto *does not know* the behaviour of the logarithmic function.

The picture presented by the last two examples is somewhat different. Here, the content of the mathematical fact, the method of solution for this or other similar problems, is known to the student; however, this knowledge is caged in a completely frozen, and inflexible outward expression. Every attempt to substitute this outward expression with another equally valid or even better one, will lead the student to loose connection with the content of the mathematical fact involved here. So, sometime randomly chosen, and often not skilfully selected an outward expression, a notation, or a picture, becomes the unifying link with the mathematical fact in student's cognitive process. This link breaks down as soon as this randomly chosen outward expression is replaced with another similar expression.

We see in all these cases of manifestation of the formalism, that its main characteristic is its unjustifiable dominance which it has over the content of a mathematical fact in the cognitive process of students, and which is represented through an accustomed, outward, and superficial expression (verbal, symbolic, or graphic) of a mathematical fact. Such a dominance is not only unjustifiable from a standpoint that in a normal process of research the content of an object being investigated, ought to be the main goal of the attention of research, but also from the standpoint that an outward, superficial expression which is in a formalistic way attached to the object, appears to be chosen at random among a large number of such equally possible choices. By attaching such a randomly chosen outward expression to a mathematical fact, and by so doing to subordinate the content of the fact to this randomly chosen outward expression, will lead to an unreliable and unstable knowledge of this fact in minds of students.

We saw that such an elevation of an outward expression in various examples, may have various consequences. Sometimes an outward expression substitutes the essence of the content, where the later completely falls out from the mind of students; sometimes it acquires an unjustifiable control over the content which it describes. In the foundation of all what we have here lies one and only one cause which we have identified earlier and in which we find the essence of the formalism of the mathematical knowledge.

In order to conduct a successful struggle with formalism, it is necessary to thoroughly avoid mixing this vice with other prevailing deficiencies in the mathematical preparation of students. In particular, we often mix the formalism of the mathematical knowledge with the phenomenon of separation of the mathematical theory from its applications. This later phenomenon, also prevalent in our school education, appears to be, similarly as formalism, the most serious deficiency in our school education. Sheer mixing of these two deficiencies, without a critical understanding of their essential differences, would bring only a harm in the struggle against both deficiencies. Therefore, it is necessary to conduct an exhaustive analysis of the relationship between the formalism and the absence of applications of the mathematical theory in the preparation of our students.

This fact, with a few permissible simplifications, can be described as follows. In mathematics, like in any other science, its primary source of knowledge, its *first level*, is the outside world, the objective material reality; the abstractions of the objective reality's relationships and forms, that is, mathematical concepts and structures, form in the edifice which is called mathematics, its *second level*; finally, used by mathematics in order to perform scientific analysis, there are outward descriptions of mathematical facts and laws together with the whole arsenal of the formal-symbolic statements which are all precisely defined with their clear verbal expressions, and which are all forming the *third*, outward-formal *level* of the edifice called mathematics. This first level is the *source* of all mathematical research; the second level forms an authentic object of research; the third level provides *tools* for this research. A separation of the theory from applications shows the severance between the first and the second level—the severance of mathematical research from its living source—the material world. Quite contrary, the phenomenon of formalism which we discussed here, represents the break-up of the rightful connection existing between the second and the third level. Here we see that tools of research cease to be just that, and they become the goal to themselves, an original object of research, more or less emasculated. By learning and remembering the outward, superficial, formalistic, or symbolic expression of a mathematical fact, this very fact either becomes absent, or is present only formally, but never connected with its content, in the cognitive process of students.

In this manner, both phenomena—the formalism, and the lack of connection between the mathematical theory and its applications, mark the broken down, normal connection which exists between various levels in the chain forming the earlier described structure of mathematical knowledge. However, the breakdown in the chain in these two cases, occurs at the chain's two different places. While the breakdown in connection between the theory and its applications represents the breakdown between the first and the second level of the edifice, the formalism of mathematical knowledge, which represents the right relationship between the two first levels, and which is the second level of the edifice carrying the mathematical content, is improperly dominated by the third, outward level of the edifice. It goes without saying, that by confining the attention of students to the outward expression of mathematical facts, we divert them from these facts. Hence, the formalism in an indirect way, by directing all mathematical preparations of students towards the practically passive third level, and which is severed from the second, renders them completely out of touch with the first level—the material reality. Although in a *spontaneous way, the formalism* is nothing but a disconnect between the outward expression from the *mathematical* content of the corresponding fact, it is neither a disconnect from its *material* interpretation nor its embodiment. Hence, in order to successfully organize a struggle with formalism, it is necessary to thoroughly avoid mixing it with a spontaneous disconnection which exists between the mathematical theory and its practical application.

Sometimes one encounters much more difficult misunderstanding when trying to apprehend the essence of the formalism. Hence, from time to time the formalism of mathematical knowledge is confused with the rigorous, logically formal method of deduction, which is necessary at all levels of mathematical science. As a consequence, this confusion reduces the struggle against the formalism to sheer elimination from teachings of mathematics the rigor of its logically formal statements. Here we have a crude vulgarisation of the problem which is a consequence of a similarity in sounding of words "formalism" and "logically formal," and which should hardly be worth of our attention. This would be analogous as if in the struggle against the *idealism*, one would eliminate from school curricula all teachings about *ideas.*

Even cruder examples of this nature one finds in articles in which under the banner of the struggle against the formalism, many accusations are presented against the mathematical science itself as taught in schools. The mathematical science is blamed for the abstract character of its concepts and for its legitimate structure. Such attempts, with avoidance of bad methodological and pedagogical errors, ought to be curtailed once and forever.

Following Engels's classical definition, to which the contemporary science has nothing to add, the object of research of mathematics are quantitative relations and spatial forms of the material world. These relations and forms constitute contents of mathematical concepts, concepts such as a number, an equation, a function, a limit, a point, a straight line, an angle, a triangle, a circle, etc. To the laws of the material world, there correspond in mathematics, abstract, logical relationships between mathematical concepts—the mathematical truth—which are expressed through axioms and theorems. In such a way, the basic concepts of mathematics and their mutual relationships, are formed by the process of abstraction from quantitative relations and spatial forms which exist in the real world. Conversely, conclusions derived in mathematics find their interpretations in describing properties of objects of the outside world, as well as in controlling the behaviour of such objects. All this clearly reflects the unity which exists between the theory and applications of mathematics. Quite contrary, the internal development of mathematics, its logical developments of concepts, its rightful process of deduction, can proceed on an abstract plane, set apart from the primary basis on which this development is built and which exists in the material world. Sometimes in order to force the connection between the mathematics and the material world on all levels of mathematical reasoning, the state-of-art of mathematics has been blamed for "separation of theory from applications." In this accusation, one sees nothing but a gross vulgarisation of Marxist principles. With regard to this type of vulgarisation, Engels made the following observation with extreme clarity: "in order to investigate the quantitative relations and the spatial forms in a proper manner, it is perfectly necessary to separate them from their contents, living the later aside as something irrelevant."

After this analysis concerning the essence of the formalism, we ought to direct our attention to the question addressing the causes of this phenomenon. These causes may have roots either in the character of the curriculum, (that is, in *what* we teach students), or, in the method of teaching (that is, *how* we teach them). I think we all agree, there are causes at work here belonging to both of these groups, and the only argument here concerns the comparative sizes of these groups. I believe that basic causes which are at work here, are clustered around the content of the curriculum, whose choice has the tendency to stimulate the formal character of learning independently of the methods used in teaching. It may be that here the text of our curricula does not play as an important role as the traditional interpretation of this text which puts too much emphasize on one matter (formal), against the other matter (content); but then again, even an unambiguous text content of the curriculum has an enormous impact here. As the methodology of teaching is concerned, in many cases it helps in promoting the formalistic tendencies, although its role in this development is perhaps lesser than the role played by the curriculum. In order to provide the argument for this opinion, I'll present here a series of more important, according to my view, situations concerning the curriculum as well as traditional methodological presentations—situations which are all capable to generate and to enhance the formalistic character of student's knowledge and their learning habits. These situations which I am presenting ought to serve only as examples whose list is by no means complete.

1. Even a cursory look at our curricula, reveals that many topics in them are without a clear purpose; this defect is so prevalent in programs, that there is no a method, by whose skilful application in teaching, this defect could be overcome.

This uniquely concerns the course on algebra, and partially the course on trigonometry. It is well known how in teaching the course of algebra, a great emphasis is put on teaching the topic called "algebraic transformations." There is no doubt about it that a smooth manipulation of algebraic transformations ought to be a part of basic necessary skills possessed by every student. However, we have to conclude that it is customary in our schools, that after the introduction of algebraic transformations, during many months thereafter, students are required to perform them on a daily basis. And while they are performing these transformations for the sake of transformation, nobody ever tells them why is all this necessary. Thus, to no end, students are factoring polynomials, even showing quite often ingenuity in this process, while at the same time they don't know the purpose for doing it. There is a very little in the whole chapter on algebraic transformations which points out to the clear objective for performing these transformations. It is clear that each such expression can be transformed in many ways; however, it is completely unclear in which of many possible ways, one has to "transform" a given expression. Also, it is completely unclear why one, and not the other way of transforming the expression was chosen. Needless to say, that while performing these transformations, students don't understand why they are transforming given expressions. The same lack of student's understanding, although up to a lesser extent, we find in proofs of trigonometric identities.

Furthermore, in school curricula one finds the almost futile introduction of complex numbers. Historically looking, these elaborate, and in student's minds paradoxical numbers, have through great efforts and a hard struggle paved their way, and they won the struggle only when due to their expediency, it became necessary to include them in the set of objects dealt by mathematical science. However, in school curricula the knowledge of this necessity stays beyond the comprehension of students despite of all our teaching efforts, and so the complex numbers will be for them an intricate part of the curriculum whose expediency they can not grasp.

How about the binomial formula? A short and simple expression $(a + b)^n$ for some reason has been transformed, by using very complicated arguments, into another, long, cumbersome, and difficult to remember expression, which students have to remember in all details. In order to perform this transformation, they have to study the whole abstract and difficult topic—the combinatorics. (This topic has no any other connections and applications in the framework of the school curriculum.) Moreover, there is no a single application of this formula in the entire school curriculum, and students who do not plan to study the higher mathematics, will remember it throughout their lives as a bright example of wasted major efforts.[1] Then, why should surprise anybody that the school teaching of mathematics is a fertile ground for breeding of formalism? Can we expect in the best possible circumstances of teaching, that students will solidly learn the essence of mathematical concepts with their laws and procedures, while the purpose of these

[1]It would be better, according to our opinion, to replace this formula in the school curriculum, (or, keep it along), with the elements of Probability theory—a living, and from a formal standpoint, a simple matter, which will provide a natural area for applications of many formulae of combinatorics and other applications as well.

concepts is unknown to them, they do not generate an independent interest, do not impress directly by their importance, and at the same time in the school curriculum they are without any important connections and applications? Obviously, thinking in this way would be contrary to basic laws of psychology of learning. Even we, research mathematicians, know quite well based on our personal experiences, that of all what we have learned in the past about scientific facts, the most of that knowledge which is left in our memory represents the recollection of their outward, formal expressions. Hence, it is even less possible to firmly anchor in the mind of students mathematical facts which according to their places in the learning program, do not generate student's interests, and which students are unable to use in a constructive and purposeful manner. In this case, student's learning process follows an inevitable psychological path of the least resistance: they memorize only the outward, formal expressions, while skipping out what they think are irrelevant, but scientifically legitimate facts.

2. During the last few years some of the most important topics from algebra which were previously scattered over various parts of the curriculum, were singled out and presented as independent topics. Everywhere one can find lessons on the functional dependence, on theory of inequalities, and on investigation of equations. This selection was made as a consequence of the acceptance of importance of selected topics, with a clear intention of making the studying of these topics more systematic, and with a hope that students will acquire a firm and profound knowledge of them. However, it appears that the effect of the intended reform was counter productive. The reasons for this were convincingly reviled in the dissertation of one of our better experts on methodology, I. F. Sludski. As it turned out, our teachers en masse understood this selection as the prohibition for mentioning these singled out topics in any other part of the curriculum. So, what happened? We all know up to what extent almost all parts of the elementary mathematics, being endowed with, and dedicated to promoting the concept of the functional dependence, gain in understanding, in visual representation, in specific clarity, in effectiveness, and in attractiveness. But here: the equation of the first order without the linear function, the equation of the second order without the quadratic function, the logarithm without the logarithmic function, the generalized exponent without the generalized exponential function; even in trigonometry, the concept of a trigonometric function has been emasculated to a great extent. This removal of topics from our curricula, which carried the breathe of the life of programs, their living and specific dynamics, the removal of topics which did not compromise with any stifling schemes, this sinister move with its inevitable consequences, has led, (and it is still leading), to a significant infusion of the formalism in teaching of algebra and trigonometry courses. In short, this removal has promoted the formalization of student's mathematical knowledge.

The same applies, although to the lesser degree, to the treatment of the theory of inequalities, and to the study of equations. Again here we have two sets of ideas, whose presence in any part of the curriculum will give to it a significant specificity and meaning. Among these we know of examples where our teachers consider it improper to use signs $>$ and $<$ in the VIIth grade, (which could be used quite usefully already in I–II grade), with a justification that "inequalities will be taught later." When dealing with the study of equations, methods of solving them ought to be considered in every part of the course. The extent to which such an approach helps in promoting the increased level of specificity, and in making attractive the process of solving equations, is best demonstrated by the outstanding collection of exercises in algebra written by Aubert and Papelier[2], where such an approach is extended throughout the entire book.

3. Moving on now to the program of geometry, my understanding is, and which was corroborated by others concerning this matter, that we have accepted a system where a "systematic" (that is, an approach pretended to be logically-formal) approach of teaching of geometry starts with the sixth year of learning. This approach is fundamentally flawed, and in particular, it will promote the development of the formalism in student's learning. First, logically-formal proofs tie up students of their age to something for which their immaturity does not supply a mental need for these proofs, especially when objects of investigation provide an ample and clear convincing. This violence against the natural state of student's development, leads students to perceive the logically—formal approach as something unnecessary—something which is trying to explain obvious things, which is nothing but a sophistry, and which they have to learn in order to obey the school discipline, and to avoid getting bad grades. Such a state of a student, before anything else, inevitably leads to the erosion of the authority of a teacher. In addition to that, a student who is unconvinced in the necessity of all these proofs, using the line of the least resistance, will only remember the outward, formalistic structure of these facts. Even we, research mathematicians, know all too well, how difficult it is to remember proofs of the most obvious theorems, and what type of scientific culture is needed in order to accept these proof on appropriate levels. Second, our system of teaching of geometry leads students to cease seeing what is before their eyes, clearly impressive geometric figures—circles, ellipses, polygons, spheres, etc., but is forcing them during many years to wander painstakingly among the boring material which provides a slim diet for geometric imagination—such as parallel and perpendicular lines in plane, mutual relationships between lines and planes in three-dimensional space, etc. Again and again, we are faced with a situation showing a curriculum for a certain age, from which on purpose was extracted everything which could promote the lively interest for the subject, its content, and which is free of the formalism. Instead, we opted to promote material which can be accepted only by overcoming the age of students.

These are, according to my opinion, the basic deficiencies in materials forming the curriculum, and which lead to the formalization of the mathematical knowledge of students. What is needed and what is possible to be done in order to fix the problem? The answer to this question is perfectly clear from all what I have said earlier here. First, it is necessary to give a focus in a purposeful way of the parts of the curriculum where this can be easily accomplished, and where this was

[2] P. Aubert, and G. Papelier, Exercises in Elementary Algebra, translated by E. S. Berezanski, and A. O. Zingol, (1940).

not done until the present time. The best example of such a change are topics about transformations of rational expressions given in the recently published book on course in algebra by Alexandrov and Kolmogorov. Here, from the outset, it is clearly stated the purpose of all such transformations, namely, to represent any rational expression as a quotient of two polynomials. Here, a comparative simplicity of the last expression which a student sees, is a sufficient reason for performing the transformation, and in any other example he knows what he has to get, and why is he doing it. The parts of the curriculum which within the scope of the school program neither can find sufficient connections with other parts of the curriculum, nor with applications, and which despite of all methodological efforts can not be accepted with sufficient effectiveness, or because of the nature of their contents, ought to be reconsidered for the possible removal from the curriculum. In this context, I dare to recommend a sufficient mercilessness: those students who are planning to study mathematics at college levels, should be able to master there these omitted topics with an incomparable benefit, because these topics will come alive in their knowledge, and they will become tools in their scientific work, due to a large number of specific and credible applications.

Second, it is necessary that the idea of the functional dependence should permeate almost all course of algebra, the closing topics of the course of arithmetic, and a significant part of the course of trigonometry. We all know too well, that even at the college level, there is no a better course which is able to eradicate the formalistic habits toward mathematics, and to promote a lively interest for research, as the course on functions. The reason for this is clear: in the theory of functions the formal apparatus plays a minimal role; an attempt to confine one's knowledge only to an outward expression of the mathematical fact, can not make even its first step. On the other hand, the dynamical nature of the idea of a variable quantity by its very essence is suited to break through all solidifying forms. Namely, with this idea, as was shown by the founders of Marxism, the dialectics enters into mathematics; the dialectics being the best tool in the struggle against all formalistic deviations. Similarly, the operations with inequalities ought to be in the whole course of mathematics, because terms such as "greater than," and "less than," are associated in the minds of students, with specific, living, and full-blooded objects. Inequality signs and the simplest properties of inequalities could be taught at early stages of learning, and mastering of this material by students with a help of a methodologically correct approach, can be achieved. Solving inequalities with unknown quantities, and solving equations of the corresponding order, should be performed at the same time. The study of equations should be extended over the whole curriculum in connection with the need for finding their solutions. In particular, the solution of every equation which involves letters, ought to be accompanied with a detailed discussion. In this way, the solution gets a meaningful interpretation which helps to avoid that solution becomes a sheer formal expression. Moreover, it is interesting to observe that selecting examples for investigating equations for the sake of investigation, represents an ill-defined goal, since there is no, and it can not be, a single textbook which can in a comprehensive way answer the question, "what does it mean to investigate an equation?" De facto, this term in various examples means different things, and it appears to be impossible to put all this under one general setting.

At the end, thirdly, I believe it is necessary to introduce sufficiently radical change in the order of topics in teaching of geometry. In a seven year school, a course in geometry should start as early as possible, in any case in the beginning grades, and to continue throughout all seven years of schooling. The structure of such a course ought to be based solely on considerations concerning its content and pedagogy—not on logical considerations. There are no obstacles for children to learn very early in such a course the simplest properties of polygons, spherical solids, etc. It is understood that the basis for such a course, is openly stated principle of avoiding logically-formal proofs. This does not mean that proofs have to be completely avoided; they ought to be introduced cautiously and gradually, and only at places where students are able to observe the need for them. It is obvious that in order for this to work, it is necessary to have a profound preparation and a pedagogical tact. So, at the beginning, logical proofs are completely absent, and later, they will appear, at first sparsely, and then more often. I believe that not earlier than in the VIII grade, the so-called "systematic" course in geometry can start, where all statements are logically deduced. I don't doubt that in such designed course in geometry, (which, by the way is not a methodological novelty—it is often found in schools abroad), difficult occurrences of the formalism in geometrical knowledge of students will be less frequent than we find them in courses now.

Let us move on to the question, which of the present methodological traditions are promoting the development of the formalistic trends, and in which way they ought to be changed in order that students maximally acquire the knowledge of the content and the sense of reality. Something of an answer in this direction, follows already from what has been said here. Too often, while students are learning various mathematical operations, we pay a very little attention in making the goal of what students are learning to be clearly understood by them in the process of learning. In those cases in which the goal as well as the content of the topic both appear to be understood by students according to our judgement, we relax our attention and do not find it necessary to keep underlining the goal of the topic while students are working on a particular problem, or proving a particular theorem. Here, it is necessary to keep repeating over and over again, the importance of a theorem being proved in a general context of the goal, its relationship to other already established concepts, theorems, and problems. While doing all these, one shouldn't spare neither the time nor efforts, because the presence in the student's knowledge of the clear understanding of roles and places played by different links of the theory being studied, will strongly promote student's understanding and recall of the contents of these separate links. To understand "why, and for what," is already a strong inoculation in the struggle against formalism.

However, the important thing lies elsewhere. Were we all to carefully analyse our personal life experiences, we would have all agreed, that what we really remember with a great regularity, are those scientific facts which were once either our main object of studies, or tools in our work and our research activity. A book or a monograph, even read carefully thrice, will inevitably be soon forgotten, if its subject was considered passively, and its content was not of importance in our own scientific work.

Personally, over many years of experience, I have developed the following way for studying a new material. If I am interested to understand the essence of a paper, not only to formally recall it, after I have read it, I'll leave it aside, and with a pencil and paper I'll try to reproduce it. In so doing, I'll try to substitute the author's way of thinking with my own way, by introducing a new approach which I consider to be more suitable. In this endeavour, I may re-phrase the statements of theorems, and often I would separate parts of arguments by creating several additional lemmas. After that, if I have been successful with described 're-formulations', I would start to think about new problems which may be generated in connection with results of this paper. All such problems which are generated in my imagination, I would with a great thoroughness write down in a form of questions, whose answers I would seek until I figure out the level of difficulty for each posed problem. Only after I have finished all these work, I shall get some assurance that the content of this paper will become in my mind a suitable, applicable tool in a moment when I need it, and not a dead weight formally accepted but totally useless.

Finally, I don't want to say that this is a way as how students should study topics in the curriculum. I gave the above detailed description of my way of studying only to show that even in the mind of a learned scientist, only those things are firmly in his mind, on which he is actively working. It is quite clear that the same holds true for the mind of a young, uneducated student. Of course, a student does not know this, he does not have enough of life experience, and in a "black book" we have to blame not him, but his teacher. We find some diligent student who is reading many times over the same text, and who is trying with great efforts to remember what he is reading. After awhile he knows by heart the text, but only to find out that after a week or so, he has forgotten everything essential in it, except a few dead phrases or formulas. And we are committing a pedagogical crime if are not guiding him, with a due pedagogical tact and knowledge, in mastering of a given material. All our pedagogical efforts here have to be directed in preventing this student to do sheer memorization of material, by showing him that such an intense, saturated, and assiduous way of learning, does not give him anything except a vague perception of the subject. He has to learn that the only way for acquiring a new knowledge, is through a process of investigation, which is an intellectually active process, and the only one which guarantees that his acquired knowledge is not of a formal nature.

The central problem which we encounter here, in my opinion, is the problem concerning the methodology for implementation of all findings which we have mentioned here. And this is a problem for every school subject. I do not intend here to find its solution. The solution of this problem can be found only through a hard work of many thinking experts on methodology, and the effort will take many years to be accomplished. However, there are several pedagogical means, which we are persistently ignoring, and which never the less I believe could significantly help with their wide applications in finding the successful solution of this problem. I want to talk about these means.

We have to use all our efforts, even to pay a great attention to the most insignificant details, all in order to stimulate, sharpen, and further induce every independent attempt of students to study the subject. A student has to acquire a habit to select a notation which is different from one found in the textbook, or used by the teacher; or, to draw a picture different than one given in the textbook. All these we can require that students do right now. Some of our better teachers are requiring this from their students. Also, we have to welcome and encourage students when in their own ways they paraphrase definitions and theorems. (Provided, of course, that such new phrasing is equivalent to the original). In general, whenever students introduce their innovations in solving problems, such endeavours ought to be welcomed in the front of the whole class as significant contributions. We all agree here that this is a right thing to do, however, in practice we are very far from it! We find often cases where not only a student is forbidden to give a different proof of a theorem from one which is in the textbook, but a teacher is forbidden to do so as well. In our schools as a rule teachers require from their students to solve all problems related to a given topic, in the same "stencilled" way, and every originality demonstrated by students in finding different ways for solving problems, is severely discouraged.

There is no doubt how much damage is done by the prevalent tradition for standardization of notations. It will suffice to think, in order to get the picture, as how far we would progress in the struggle against the formalism in solving equations, were our students as fluent in solving a system of equations with unknowns a, b or k, l as they are with unknowns x, y. Were I to write a primer with exercises in algebra, I would use different notations for unknowns in different equations. And we have books on arithmetic in which a topic is entitled, "exercises with x". For any mathematically cultured ear, such a title would sound as a calamity of great vulgarity. Likewise, it is not necessary always to denote by a_n the general term of a progression; use t_r and u_k as well.

We should require as little as possible from students to learn anything by heart. It is nice to know poetry by heart. In mathematics, knowing by heart definitions and theorems is justified only in early grades. However, as soon as students reach the level when they can express anything in "their own words," it is not only necessary to tell that they have a right to do so, but it is our duty to make them to do it. Let us conduct the following experiment in a single class: allow those students who can on their own, to paraphrase definitions and theorems, and allow those who can't do it, to learn statements of definitions and theorems by heart as they are written in the textbook. By praising the achievement of the former group as the right thing to do, we shall create a healthy and useful competition among students in this class. In particular, it is very difficult to make students learn by heart those "definitions" which are not essentially definitions, and which describe nothing ("definition" of a number, of a point, of a line, of an angle, etc.).

A reader may doubt the effectiveness of all these small steps, such as changing notations, paraphrasing definitions, and theorems, etc., in improving student's learning? I think that by motivating students by these "small steps" to work creatively on their own, represents the first step in helping the teacher to generate student's interest for the subject, and which is in itself a sufficiently important achievement. These "small steps" help in building student's character, and they help them in forming habits as how to answer, by figuring on their own, any posed question. A student may correctly recite a definition learned by heart, without understanding of what is going on there. However, if called to present a definition, or to describe a concept in his/her own words, it will be impossible unless a student has a clear understanding of the essence of definition/concept in question. A student who knows how to prove a theorem in geometry by using any other drawing different from one presented in the textbook, by so doing shows that he/she has understood the theorem's essence. In all these cases, we have made already not-so-insignificant steps in the struggle against formalistic tendencies.

The second important tool in the struggle for promoting contents of mathematical knowledge, has to be a change in the nature of our examinations. Questions on exams ought to be formulated in such a way, that their correct answers can be given only by knowing the content of a concept in question, and not its formal description. That this makes sense, is easy to accept, and easy to implement. Let me clarify this with the following example. A few years ago I attended the graduating exams in algebra in one of Moscow's high schools. A female student's question was to derive the binomial formula. She wrote on the chalkboard a long chain of equalities. Her teacher, after a cursory look at what she has written, said: "this is correct, you may go!" Then, I jumped in, and asked her to explain how she got the second equality from the first, the third from the second, etc. She couldn't answer this question, although I gave her the ample time to prepare the answer. I have a reason to believe, that this type of formalistic approach as demonstrated by this case, and which is quite typical, is not present only on exams, but at lectures as well. Does one need to tell that this necessarily leads to the formalization of the mathematical knowledge! It is clear, were students to know that both, in the classroom and on the exam, they have to demonstrate the knowledge of the content, not only its formal description, they would have been stimulated to learn contents of the material. Always and everywhere the exam questions ought to be selected to clearly show by student's answer, whether he/she really knows what is he/she talking about, or, that he/she is only writing down a series of symbolic expressions, or, repeating verbal statements, which all have been learned by heart. It is very important that students should know in advance that exam questions are going to be of this nature, and what they are expected to know in order to answer them.

It is left to me to remind the reader about what I have said here at the beginning: consider my modest effort here only as a first contribution toward the enormous and difficult task in understanding the essence and sources of the formalism in the mathematical knowledge of high school students, and what has to be done in the effective struggle against this basic deficiency in the mathematical preparation of high school students. I am unable, and I don't want to present anything definite here. I hope that what I have presented here will provoke a large number of critical comments, which will in turn lead to the solution for overcoming the difficult problem of the formalism, and by so doing will radically improve the quality of the mathematical preparation of high school students.