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A BROADER WAY THROUGH THEMAS OF ELEMENTARY SCHOOL MATHEMATICS, III

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Abstract. In contrast to an existing tendency, the teaching procedures related to the block of numbers 1–20 are considered with nicety of detail. Interdependence of addition and subtraction is treated with a differentiation of syntactic matters from those semantic. A comparison of matching and counting is also included in order to show why matching alone would not lead to the building of the system of natural numbers.

AMS Subject Classification: 00 A 35

Key words and phrases: Block of numbers 1–20, interdependence of addition and subtraction, matching and counting.

8.2. Block 1–20. The range of numbers 0, 1, ... , 10 plays the fundamental role of generators for the whole system **N**. Namely, each natural number is uniquely represented as a polynomial of the form

$$a_n 10^n + \cdots + a_1 10 + a_0,$$

and, when this polynomial is shortened to be a positional decimal notation, it is written as $a_n \ldots a_1 a_0$. The involved coefficients $a_0, a_1, \ldots, a_n$ belong to the range 0, 1, ... , 9. When addition or subtraction is performed, it goes according to the rules of operating with polynomials followed by possible carryings or borrowings which keep the set of coefficients within the range 0, 1, ... , 9. And when these performances are carried out on the decimal notations, the whole calculation is reduced to a series of sums and differences which are listed in addition and subtraction tables. Modern practice aims at a spontaneous memorizing of these tables accompanied with a complete understanding of how the included sums and differences are gradually obtained from those which are smaller or easier. Since the realization of this aim goes through the range of numbers 0, 1, ... , 20, already by it, these numbers form an important didactical whole, called the block 1–20.

8.2.1. Extension of the block 1–10. Copying from adults and other children, many preschoolers learn to recite in order the names of numbers up to ten or twenty. Also, they can attach these words to the groups of objects in their everyday surroundings. In the specific atmosphere of a class, children spontaneously and by themselves, develop this ability still further. But here we consider the role of counting as an important didactical step in the course of a systematic teaching process.

After completing the lessons concerned with the range of numbers 0, 1, … , 10, the teacher has to activate his/her class so to be sure that each of the pupils can count up to twenty easily. Thus, the most important matching set is extended to meet the needs of this block in extension.

In section 6 of this paper we considered the examples of two sets, when the children are supposed to find, by counting, the number of elements of each of them. Asking the question "how many altogether", we stimulate them to unite the sets (to consider them as a whole) and to find, again by counting, the number of elements of the union. On one hand, we direct them to unite naturally (and without any symbolic formalization of prearithmetic operations) while at the same time they become gradually familiar with the examples of addition scheme. The same is again on the scene here, but now with the examples of sets of cardinality not exceeding 10 and with their union, whose cardinality exceeds 10. Let us consider an example.

A tray is seen with 8 cakes on it and another one with 7 cakes on it. How many on each? The answers are 8 and 7. How many cakes there are altogether? Children will count (or count on from 8) and they will answer in words: f i f t e e n c a k e s.

According to the already established meanings related to the block 1–10, children know the numbers 8 and 7, the sum $8+7$ and now they name the resulting number using the word "fifteen". A series of similar examples completes the didactical step we mentioned here.

The second step in this extension starts with the sums

$$10+1, \quad 10+2, \quad \dots, \quad 10+10$$

and with their abbreviations

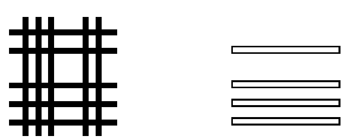
$$11, \quad 12, \quad \dots, \quad 10$$

which are read

eleven, twelve, … , twenty

and when, the two corresponding notations are connected with the equality sign.

In the real teaching situations, number images should be used as, for example, the following one



$$\mathbf{10} + 4 = \mathbf{1}4$$

Fig. 19

(the "blue" ten plus the "red" four equals the "two colour" fourteen). Such a usage of digits in colours suggests the positional values implicitly.

Up to this point, each number from the range 0, 1, … , 20 has its own notation and, in addition, a group of easy sums has been established as well.

8.2.2. Cases of finding sums and differences reducing them to smaller and easier ones. Casting a glance at some contemporary text-books in arithmetic, one can observe a complete negligence of the cases of summation and subtraction which cross the 10-line, although it was a core thema of earlier authors of such text-books. It is true that, thanks to the machines which do them to us, the lengthy calculations with mechanical speed and accuracy are no longer regarded as an essential part of arithmetic teaching. But this does not mean that the calculations related to the initial number blocks should be left out. On the contrary, their systematic treatment which provides meaning, skilful performance and right amount of drill is a permanent didactical task. Neglecting such a treatment, some authors expect children to "find their own way to the results". Their favourite means often are word problems, believing that the embodiment of meaning into physical situations helps the calculation. Reacting to it, we take the freedom to make a figurative definition: calculation is the way of finding the more complex from the more simple.

In the frame of this topic, the simpler sums are the smaller ones whose values do not exceed 10, as well as the already mentioned easier ones: $10+1$, $10+2$, $\dots$, $10+10$. The simpler differences are those smaller, with the minuend not exceeding 10, as well as the following easier ones: $11-1$, $12-2$, $\dots$, $20-10$. The more complex sums and differences are those which cross the 10-line.

The methods of adding up over ten and subtracting (taking away) down below ten are the climax of the first grade arithmetic. Their application runs through an active and motivated operating with arithmetic expressions. It is also the place where the rule of association of summands and the rule of subtraction of sums are applied with a right motivation. Moreover, these methods are the ground upon which the children build up the addition and subtraction tables, finding the results with a full understanding and memorizing them spontaneously.

Sized properly, the operating on arithmetic expressions links arithmetic to algebra. At that stage formed operating skills are easily transferred to algebra and the established meanings are a semantic ground without which algebra would be a mere play with letters. Therefore, negligence of these methods and no effort to set them work is an evident sign that some authors have overseen something very essential.

Though we often do it, we will not enter here into the details of a real teaching process, because such contents are well known and they are not subjected to essential variations.

8.2.3. Some teacher's uncertainties. A well planed usage of arithmetic expressions from the very beginnings of the teaching process is a remarkable innovation. But a number of teachers, particularly those from elder generations, encounter some problems when treating such a matter. I can remember some of them who were perplexed with the fact that, once we say that the sum of numbers 5 and 3 is $5+3$ and, the other time that it is 8. Which way is correct, was their question. Explaining, I first took a simpler example.

When we see "8" written on the blackboard, once we say it is the figure eight, the other time it is the number eight. In the former case we recognize a sign—something syntactic and in the latter case we also attach to the sign its meaning—something semantic.

In the same way, when we see "5 + 3", we recognize something which has been set down in writing and what is a collection of single signs, technically called an arithmetic expression, more specifically, a sum. When signifying, the sum is a syntactic concept. When we also attach to such a notation its meaning, then it represents the number eight and, in this sense, the sum is a semantic concept.

When two different arithmetic expressions stand for the same number—have the same numerical value, they are called numerically equivalent and then, they are connected by the equality sign. In respect to it, the equality is an equivalence relation in the set of all arithmetic expressions and calculation can be seen as a chain of simpler and simpler equivalent expressions linked by the equality sign, which ends with the most standard one in the form of a decimal notation.

Finally, such and similar uncertainties of teachers make us believe that the instructors who teach in and the authors who write books for elementary schools should be acquainted with the rudiments of mathematical logic and a chapter from the Kolmogorov's book [14][1] might be a good inspiration for it. Such contents would, prior to all, help the elimination of an existing jargon in schools which is due to a long generation of obtrusive preachers.

8.2.4. Linking addition and subtraction. We can read in many a book on teaching matters that subtraction is the inverse operation to addition or, more symmetrically said, that two operations are inverse to each other. To understand what is meant by it, one thinks of the real world situations which call for addition and subtraction. For example, giving is opposite to taking and vice versa. But considering these two operations more abstractly, it is probably better to say that they are related to each other. And this means exactly that the three equalities

$$m + n = s, \quad s - m = n, \quad s - n = m$$

are at the same time true or false.

Now the question is how to transpose it didactically and, so embodied, to communicate to children. First, of course, by the use of examples from the natural environment (including their pictorial representations). Having such an example of situation fixed, the requirements are varied so that the children are induced to write the equalities $m + n = s$, $(n + m = s)$, $s - m = n$, $s - n = m$, while the situation keeps still. Then, as a second step, schemes of the form

[1] References numbered 1 to 13 are included in the first two parts of this article (this Teaching, vol. II, 1, p. 58, and vol. II, 2, p. 103)

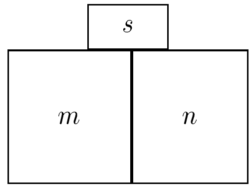


Fig. 20

should be used and the children engaged to formulate some word problems which call for each of the above equalities to be written. And finally, groups of more formal exercises as, for example, the following ones

a) $3+ \_\_ = 9$	b) $8-5= \_\_$	c) $7- \_\_ = 3$
$9- \_\_ = 6$	$5+ \_\_ = 8$	$\_\_ +4 = 7$
$9- \_\_ = 3$	$\_\_ +5 = 8$	$4+ \_\_ = 7$

should be given. As far as a child, after having the first equality completed, finds the other two left to be easier, his/her acquisition of this relationship is evident. To control the degree of acquirement of such knowledge, the teacher may prepare even more formal exercises of, for example, the following type: When you check that $8+9=17$, then, without checking, you can write: $8= \_\_ - \_\_$, $9= \_\_ - \_\_$, or else: By calculating, you complete $17-8= \_\_$, then without calculation, you can write $8+ \_\_ = \_\_$ and $9+ \_\_ = \_\_$; and so on.

When adding we write, for example: $5+3=8$ and when decomposing into summands: $8=5+3$. We see, thus, the symmetry property of equality expressed in writing. But, without a specific motivation there is nothing which would move us to reverse such an equality. Just the opposite, in practice a kind of asymmetry is preferred: a more complex expression is usually seen on the left-hand side of an equality or all terms containing an unknown quantity are taken to the left-hand side, etc. And when no attention is paid to this property, some secondary school students may be found who continue to solve the equations as this one: $-5=x$, $-x=5/\cdot(-1)$, $x=-5$.

On the other hand, equality in the set of numbers is an identity relation and, since each thing is always equal to itself, the symmetry property is a matter of course. More generally, it is the same with any equivalence relation if defined by equality of certain properties (say, in the set of expressions to be equivalent means to have the same numerical value). Therefore, the problem of treatment of symmetry property should be seen as a matter of the use of syntactic apparatus.

Using space holders and by other means, children should be induced to write all eight equalities which are associated with the situations represented by the scheme in Fig. 20. Attaching to a fixed situation the equalities

$$m+n=s, \qquad n+m=s, \qquad s-m=n, \qquad s-n=m$$
$$s=m+n, \qquad s=n+m, \qquad n=s-m, \qquad m=s-n,$$

in that way they procedurally express the link between two operations as well as the symmetry property.

If, happily, learning of lengthy calculations is no longer accentuated in the contemporary school, the cultivation of pupils to use some pieces of syntactic apparatus correctly and with accuracy is a new didactical task which is motivated by the prolongation of educational process.

8.2.5. Comparison of counting and matching. As already said (paragraph 6.2), the ideas of natural numbers result from a process which consists of a perception of concrete sets and an abstraction described as the ignoring of the nature of their elements and the way they are arranged. In the case of small sets, the corresponding mental images exist formed in our mind and, for instance, we are able to immediately differentiate a set having three elements from another one having four elements. But, as known, this ability of ours is very limited. Perceiving two larger sets of objects, the corresponding inputs cannot be interpreted so sharply as to distinguish them according to the number of elements. Excepting the case of a few initial numbers, the mental apparatus contains nothing but a rather fuzzy inner representation to which the world "many" is attached to express cardinality of the sets. In this respect we are not far from the prehistoric man who counted up to five only. (To those who doubt it, we suggest to try to imagine clearly a monotonous linear ordering of ten dots.)

If a set does not match with a mental image in our mind, it may do it with another set. Given two sets A and B, if for each element a in A there corresponds an element b in B so that for different a's, different b's are corresponded and vice versa, then we say that these two sets match. For such two sets we also say that they are mapped one-to-one on each other. Pointing out the common property of two sets which match, we say that they are equipotent or, less technically, that they have the same number of elements. The last phrase could easily provoke the question "which number". And, in this theoretical setting, the answer would be "the number which each one of the sets itself represents". Thus, the concept of a number (technically, of a cardinal number) is defined to be the common property of all sets belonging to the same class of mutually equipotent sets (technically, such a class, itself).

Now our curiosity has been arisen and we are eagerly expecting to know what according to this definition, would be our familiar numbers 1, 2, 3, ... And they are defined as follows.

The number 1 (2, 3, ...) is the common property of all sets equipotent to the set

$$\{x\}, \quad (\{x, \{x\}\}, \quad \{x, \{x\}, \{\{x\}\}\}, \quad \ldots).$$

Defining numbers of the sequence 1, 2, 3, ... , a sequence of sets formed as combinations of an element "x" and the curly braces is used. Instead of this sequence, many others may equally be used. For example, replacing, member by member, each set with an equipotent one, the following sequence of pictorially represented sets is obtained

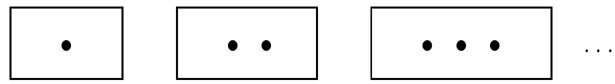


Fig. 21

As is easily seen, to define a natural number, say 3, a three element set is used what, in that way, brings out the intuitive meaning of this number expressed by a significant sign. To define all natural numbers, an unbounded sequence of such signs would be used. Since a complete realization of such a sequence is impossible, the symbol " ... ", read "and so on", carries the command "continue doing as it has been done". For example, to continue the sequence of pictorial signs in Fig. 21, the next step is to draw four dots, the following five and so on, at each new step, a dot more is added. A third sequence which can be equally well used consists of sets of decimal notations

$$\{1\}, \quad \{1,2\}, \quad \{1,2,3\}, \quad \ldots$$

or of their verbal substitutes

$$\{\text{one}\}, \ \{\text{one, two}\}, \{\text{one, two, three}\}, \ \ldots$$

All these settings of symbols, one after another, or their possible consecutive reciting could remind us of counting, what it essentially is, but in a different form. Therefore, we see counting as an action inevitable in the process through which the system **N** of natural numbers is constructed. This construction goes divided into portions (number blocks) with counting also converted into addition and multiplication. Thus, a reasonably large range of numbers is formed within which, due to associations with nice groupings (in ones, tens, hundreds, etc.) each number also has an intuitive meaning. Out of this range everything comes as a result of the "and so on" effect. We can really see this effect at work in many situations when something is demonstrated in a number of cases, and, as soon as a rule manifests, "and so on" is added to mean that the same demonstration goes in all other cases and, in fact, it is nothing more than a spontaneous form of the fundamental mathematical principle called total induction. As well known, its explicit formulation reads:

Given an infinite sequence of propositions $p_1, p_2, \ldots, p_n, \ldots$, *if*

(I) p_1 *is true;*

(II) p_n *implies* p_{n+1} *for each* n,

then all these propositions are true.

In its spontaneous form, this principle has been known from the times of Euclid. A first explicit formulation appeared in the book "Arithmetica" written by Italian Renaissance mathematician F. Maurolycus. Acknowledged and effectively used as a method of proof by Pascal, the explicit form of this principle is often associated with his name.

A deep treatment of the induction principle can be found in Poincaré's book [15]. Chapter XI of Freudenthal's book [3] is a brilliant essay on comparison of counting and matching.

Attaching an addendum to this paragraph, we want to show in a logically more sharp way that the number system **N** is characterized by counting (and from it derived operations) together with the property which the principle of induction expresses.

Addendum 6.

Let S in the triple $(S, 1, s)$ be a nonempty set, 1 a selected element of S and $s\colon S \to S$ a one-to-one mapping. Let the following axiom holds.

(I) *for each* $x \in S$, $s(x) \neq 1$.

The triple $(S, 1, s)$ is an example of a mathematical structure and two such examples $(S, 1, s)$ and $(S', 1', s')$ are isomorphic if there exists a mapping $\varphi\colon S \to S'$ which is one-to-one and onto, which maps the selected elements one upon another, i.e. $\varphi(1) = 1'$ and for each $x \in S$, $\varphi(s(x)) = s'(\varphi(x))$ holds.

Let us remark that the mapping s can be seen as a way of abstract counting which begins with 1 and which produces the sequence of iterates

$$1, \quad s(1), \quad s(s(1)), \quad \ldots$$

The axiom (I) ensures that all its members are different and so makes the set S be infinite. But, apart from the members of this sequence, the set S can contain many other elements.

Letting S be **N** or the set of ordinals $(1, \alpha)$ which are less than a limit ordinal α, $(\alpha > w)$ and defining s by $s(x) = x + 1$, two nonisomorphic examples of this type of structure are obtained. A drastically different example from these both is obtained, taking for S the disjoint sum of **N** and any nonempty set X and extending s to be the identity mapping on X.

Such a weak structure becomes much more determined when the following axiom

(II) *for each* $A \subset S$, *if* $1 \in A$ *and*

$$s[A] = \{\, s(x) \mid x \in A \,\} \subset A,$$

then $A = S$,

is added.

The system of two axioms (I) and (II) is now categorical, what means that any two examples $(S, 1, s)$ and $(S', 1', s')$ which satisfy them are necessarily isomorphic. (It is easily seen that the set S has no other elements than 1, $s(1)$, $s(s(1))$, $\ldots$. Then, an isomorphism mapping S to S' is easily found.)

The example of the set **N** with s given by $s(x) = x + 1$ satisfies the axioms (I) and (II) and, hence, there exists no other example which is not isomorphic to it. Thus, the axioms (I) and (II) fully characterize the system **N** of natural numbers.

The axiom (II) is nothing more than a variation of the principle of total induction and the forerunning consideration reveals its strength.

In a varied form, these axioms are due to G. Peano (1858–1932) and called the axioms of arithmetic.

Defining addition and multiplication inductively by

$$\begin{aligned} n+1 &= s(n), \qquad & n+(m+1) &= s(n+m), \\ n \cdot 1 &= n, & n \cdot (m+1) &= n \cdot m + n, \end{aligned}$$

all familiar properties of natural numbers can be established on this basis.

8.2.6. Role of nice groupings. Numbers as shapes. If it is hard to imagine a monotonous series of ten dots, an image of two hands and the ten fingers on them, is easily evoked in the mind. What we really cause to appear are two very familiar shapes, each of which bears an intuitive idea of number five. Thinking of the number of fingers so seen in the mind, it will be $5+5$ rather than 10. A regular arrangement of ten sticks, as the one in Fig. 19, is also easily imagined and, again, as a pure shape. When number images in the form of appropriately sized sheaves of sticks representing 10 and 100 have been used, then it is easier to perceive or imagine 222 than 6 sticks in a monotonous order. Of course, imagining and perceiving of numbers are interdependent and, if not related to nice shapes, such our ability hardly surpasses the number five and we have already mentioned that fact as a reason why the range of numbers 1–5 should be treated as an exclusive didactical topic.

An object as a sheaf of hundred sticks does not reflect clearly the number 100. On the opposite, by the way of its usage, it becomes a signifying sign. A bundle of hundred one dollar banknotes has the same purchasing power as a single hundred dollar banknote. Understanding the meaning of the number 100, we could say that the latter signifies what the former is. In the sequence

$$10, \quad 10^2, \quad 10^3, \quad \ldots, \quad 10^n, \quad \ldots$$

the initial members are often used so that they reflect various meanings in the surrounding reality. For bigger n, 10^n is a huge sheaf composed of 10 smaller, each containing 10^{n-1} sticks. This is a typical inductive definition by which, in a less formal way, these powers get the meaning. And if the numbers from a reasonably large initial range do reflect life, those out of it are understood only according to the way we operate on their notations.

Two equipotent sets of objects represent the same number intuitively. One of them may have its elements nicely grouped, the other just be a chaotic heap. Ordered or disordered, it does not effect the numerical meaning they bear, but the perceptional grasp of a nice pattern may be very important for deriving suitable notations for numbers. For instance, perceiving the elements of a set in three groups of at most nine subsets so that the subsets in the first group are 1-element, in the second 10-element, in the third 100-element, we have a pattern which is the basis for a three digit notation.

The point of view of some specialists that matching of sets leads to a proper understanding of natural numbers is more than doubtful. When, as a result, performance of routine tasks is ignored, then it leads into a wrong path going aside the course established by classical educational reformers.

Our aim in this paragraph was to show that matching without counting would not produce the system **N** of natural numbers. On the other hand, depending on grouping of elements of finite sets, a corresponding system of notation is also produced. Such groupings, though not essential for establishing the concept of natural numbers—not being its invariant property, they are the basis upon which **N** is built. In the paragraph 7.1 of this paper, we considered some historical systems of notation which do and which do not produce the whole system **N**.

We include the following addendum for those readers who have a good professional acquaintance with mathematics.

Addendum 7.

As soon as an equivalence relation is given in a class of objects, a system of morphological types results from such a classification. A property is an invariant of this classification whenever it is shared with all mutually equivalent objects. Similarly, a term is invariant if it has the same meaning for any two equivalent objects. All invariant terms of such a morphology constitute its invariant language. Let us consider some examples.

Homeomorphic topological spaces are topologically equivalent and each class of mutually equivalent spaces represents a topological type. But to show that such morphological types exist, we often use a language which is not invariant. For example, n-sphere is defined by means of coordinates as the set of points in $\mathbf{R}^{n+1}$ such that

$$x_0^2 + x_1^2 + \cdots + x_n^2 = r^2,$$

and then used to fix the topological type of n-sphere as a class of objects equivalent to it.

To develop interesting systems of invariants, spaces having polyhedral structure or the structure of CW complexes are used. In the process of describing these structures a descriptive (noninvariant) language is employed and many properties of such objects are not invariant in the topological sense. We may call such structures, which are important to develop a theory, the associated descriptive structures. As another example, we can mention the way how groups are determined by presentations—systems of generators and defining relations, what is again an example of descriptive language. And instead of manipulating with isomorphism types of groups, we do it with their presentations.

There exist many other situations in mathematics which illustrate the importance of descriptive language and associated descriptive structures and without them, a theory would be deprived of meaning. Returning to the system **N**, the nice groupings of elements of sets are seen as important associated descriptive structures and derived notations and the rules of manipulation with them as an important descriptive language. Without them, arithmetic would be reduced to an entrance porch.

At the end, we express our opinion (also formed in touch with school reality) that children learn arithmetic with lightness and ease from the text-books worked out with great care and nicety of detail and not from those ones whose exclusive feature is a too much elaborate adorning.

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