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# FUNCTIONS FOR FOURTH GRADERS

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**Abstract.** Ordering is an important activity for cognitive development of a child and it has a fruitful form in the idea of (finite) sequence. A schematic representation of concrete sequences (of numbers or letters) is used to foster the idea of function. Leading pupils to discover simple laws of correspondence and to compose the expressions which formulate them seems to be the first possible instance of such activities feasible in elementary school practice.

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The Belgrade Society of Junior Mathematicians "Archimedes" organizes each year a summer school at the Youth Settlement "Mitrovac", in the Mountain of Tara. Tara is the home of the most graceful European conifer Omorika, a tertiary relict discovered by botanists in the 19th century. In such a relaxing ambient, children have some math lessons in mornings and they spend their afternoons playfully.

Three times, I had there a series of four lessons for a group of fourth graders (one combined with some third graders). In a more or less varied form, I cover the material of three first lessons in the three first paragraphs of this article. Hoping that it could be a model to be followed, my exposition is impersonal with a teacher and his/her class acting. Last summer (2001) I included the content of the fourth paragraph and I expose it approximately as it was going on.

This exposition, taken as a whole, demonstrates what a teacher could do with a group of bright boys and girls. If even attainable by an average class, such a material would be time-consuming and so too much demanding for a normal curriculum. But there is, of course, plenty of scope for inclusion as well as simplification of the content of the first three paragraphs into current school subject matter. Problems of the type "discovering a rule" have been the assignments for fourth graders in Serbian schools for several years and the teachers report that at least the better half of their classes finds them very interesting and solves them successfully.

Considering the ways of concrete representation of mappings, H. Freudenthal, [1], distinguishes displacements from transfigurations. In the former case, the domains are sets of places and the rule $f$ tells that the object at $x$ is displaced to $fx$ and in the latter, the domains are sets of objects and $f$ tells that the object $x$ is transfigured into $fx$. In the case considered here, domains are sets of places and $f$ tells that the object (number) $fx$ is in the place $x$.

For some recent ideas of early elaboration of the function concept, the reader is directed to 12th ICMI Study Conference Proceedings, [2].

## 1. Distinguishing sequences from sets

Children are supposed to know how sets of conventional symbols (numerals, letters) are denoted when curly braces are used. To revise the way how sets are equated, the teacher prepares a number of exercises.

**1.** If the equality is true, write "⊤", if not, write "⊥".

(I) $\{1,2,3\} = \{3,1,2\}$ ___

(II) $\{a,b\} = \{a,b,c\}$ ___

(III) $\{a,b,c\} = \{b,c,a\}$ ___

(IV) $\{a,b,c\} = \{a,b,d\}$ ___

**2.** Write sets of numerals with which the following numbers are written.

(I) 235, 253, 532 ... { }

(II) 441, 144, 414 ... { }

(III) 25, 52, 85, 58, 28, 82 ... { }.

**3.** Write the sets of letters with which the following words are written.

(I) pot, top ... { }

(II) ten, net ... { }

(III) tap, pat ... { }.

Here the teacher points out that several different numbers (words) can be written using the same set of letters (numerals).

**4.** Using first seven letters of English alphabet, write as many English words as you can.

Teacher writes the set on the blackboard: $\{a,b,c,d,e,f\}$. Children are let to pick up words and as they are found, the teacher writes them in groups.

| | |
|---|---|
| add, ace, abaca | bed, bede |
| cab, cabbage | dad, deaf |
| ebb | fad, feed |
| gab, gad, gag | |

If not found by children, the teacher intentionally includes longer words: abaca, cabbage.

For each group, the set of used letters is written. Then, some pairs of words are compared: which is longer (shorter), how many letters has each, in which words some letters appear twice, three times, etc.

Now the teacher takes the word "cabbage", writing

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{c} & \text{a} & \text{b} & \text{b} & \text{a} & \text{g} & \text{e} \end{array}$$

and reading arrows: in the first place is c, in the second a, ...

Then, writing

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | | 1 | 2 | 3 |
| ↓ | ↓ | ↓ | | ↓ | ↓ | ↓ |
| p | o | t | | t | o | p |

he/she asks pupils why the two words are different. Commenting, he/she should say that the words are already different because "pot" has in the first place "p" while "top" has "t".

**5.** Using the digits: 1, 2, write all

(I) two digit numbers

(II) three digit numbers.

Some attention should be paid to the procedure of forming such sequences:

One digit numbers — Written in column

1, 2

1
2

Two digit numbers

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 1 1 |
| 1 | 2 | 2 | 2 | 1 2 |
| | | | | 2 1 |
| | | | | 2 2 |

Three digit numbers

| | | | | |
|---|---|---|---|---|
| 1 | 1 1 | 2 | 1 1 | etc. |
| 1 | 1 2 | 2 | 1 2 | |
| 1 | 2 1 | 2 | 2 1 | |
| 1 | 2 2 | 2 | 2 2 | |

**6.** Why are the following pairs of numbers different?

| (I) | | (II) | | (III) | |
|---|---|---|---|---|---|
| 3 | 3 | 2 | 2 | 1 | 1 |
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| 1 1 1 | 1 1 2 | 1 1 2 | 1 2 2 | 1 2 2 | 2 2 2 |

(The teacher adds the arrows and comments: yes, the digits in the third place are different, etc.)

Now the teacher summarizes in words what has been done, saying: When we put numbers, letters, etc. in sequence, paying attention how we order them, then we get a sequence of numbers, letters, etc. Reviewing some of already considered examples, he/she intentionally uses the word "sequence" followed by naming of its terms whose number should also be counted. For instance, cabbage

is a sequence of letters, composed of 4 different letters arranged in 7 places. Let us point here that no independent use of the word "sequence" is admissible either the question as "What is a sequence?" has any sense. To say it simply, such a word should not be used separately from the concrete examples to which it is referring.

We use an example to show how a confusion may arise when the terms of a sequence are not visibly separated.

**7.** Write six sequences using each of the numbers 12, 27, 135 just once.

Writing carefully, we get

| | |
|---|---|
| 12 27 135 | \|12\|\|27\|\|135\| |
| ... | ... |
| (with spacing) | (with place holders) |

Writing carelessly, a warning confusion arises

1227135

For instance, 1227135 may be interpreted as a sequence of digits or, in several ways, as a sequence of numbers

12 27 135;  12 271 35;  1 22 71 35; ...

Now the pupils are told that, to avoid such confusion, spacing or place holders are used or even better, commas

$$12, 27, 135 \qquad 12, 135, 27 \qquad 27, 12, 135 \qquad \dots$$

Following this example, the teacher insists on the use of commas which separate terms of a sequence, respecting exceptions: commas are never used to separate letters of a word or digits of a number.

## 2. Symbol " ... "

Taken formally, a sequence (an $n$-sequence) in a set $S$ is a mapping of the set $\mathbf{N}$ (the set $\{1, 2, \dots, n\}$) into the set $S$. This mapping is usually given by a formula $n \mapsto f(n)$ called the *rule of general term*. But at this level, a sequence should be experienced as a real seriation of objects in the space or signs on paper, which come one after another. Together with the observance of an object, as a part of background, its place in the space is also acquired by the mind. Consequently, we should speak of terms being in first, second, ... place,

Now the teacher proceeds giving some examples.

**1.** Write the sequence of 7 consecutive numbers starting at

a) 25  b) 17

and going forwards. Do the same but going backwards.

**2.** Write the sequence of 5 consecutive even numbers starting at 28

a) going forwards  b) going backwards.

Do the same with odd numbers starting at 27.

(When these sequences have been written, the teacher asks the pupils to read the terms in second, third, etc. place.)

**3.** How many terms has the sequence of consecutive numbers

a) which starts with 1 and ends with 100?

b) which starts with 48 and ends with 100?

Answering b), some children are inclined to calculate this way

$$100 - 48 = 52$$

and such a mistake should not be ignored.

Here the teacher has to use shorter sequences of smaller numbers, say, 7, 8, 9, 10 to demonstrate that the number of terms is not $10 - 7 = 3$. Asking "how many missing terms if the sequence had started with 1", he/she directs the children to solve such problems correctly.

Now the teacher tells the class that when a sequence has many terms, we do not write them all and that after we have a few initial terms written, the symbol "…" is used to denote seriation. Of course, the best way to tell it is an explanation by examples.

**4.** Given the sequences

a) 2, 4, 6, … , 18  b) 10, 12, 14, … , 28

c) 1, 3, 5, … , 17  d) 11, 13, 15, … , 29

tell, in each case,

(I) which terms have not been written,

(II) the number of all terms.

Describe the given sequences (sequence of consecutive even (odd) numbers, starting with … and ending with … ).

Elaborating, the teacher writes

a)
$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & \_\_ \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ 2 & 4 & 6 & \dots & 18 \end{array}$$

b)
$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & \_\_ \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ 10 & 12 & 14 & \dots & 28 \end{array}$$

c)
$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & \_\_ \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ 1 & 3 & 5 & \dots & 17 \end{array}$$

d)
$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & \_\_ \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ 11 & 13 & 15 & \dots & 29 \end{array}$$

leaving to the children to write the number whose place is indicated by holder.

Finding the number of terms as well as the rank of the last term are not so easy assignments. Making effort to solve such problems, children will enjoy more discovering of rules which connect rank of a place with the number in that place.

Now the teacher includes a number of examples where the symbol " ... " stands to suggest that a sequence is not ending. For instance,

a) 1, 2, 3, ... b) 2, 4, 6, ...
c) 1, 3, 5, ... d) 1, 10, 100, ...

Naming: the sequence of all: a) natural numbers, b) even numbers, c) odd numbers; d) decimal units, the teacher also suggests the function of the sign " ... ". Then, he/she rewrites the sequences

b) $2\cdot 1, 2\cdot 2, 2\cdot 3, \dots$ c) $2\cdot 1-1, 2\cdot 2-1, 2\cdot 3-1, \dots$

pointing out that terms of the latter sequence are one less than those of the former. And the question of finding a rule opens in the form

$$\text{b)}\quad \begin{array}{ccccc} 1 & 2 & 3 & & n \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow \;\dots \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & & \_\_ \end{array} \qquad \text{c)}\quad \begin{array}{ccccc} 1 & 2 & 3 & & n \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow \;\dots \\ 2\cdot 1-1 & 2\cdot 2-1 & 2\cdot 3-1 & & \_\_ \end{array}$$

Planning such an activity the teacher defines so the problem and helps children understand the function of all involved notations. Now, he/she may expect the answers

$$\text{b)}\quad \begin{array}{c} n \\ \downarrow \\ 2\cdot n \end{array} \qquad\qquad \text{c)}\quad \begin{array}{c} n \\ \downarrow \\ 2\cdot n-1 \end{array}$$

and writing: d) $10^0, 10^1, 10^2, \dots$, also the answers

$$\text{a)}\quad \begin{array}{c} n \\ \downarrow \\ n \end{array} \qquad\qquad \text{d)}\quad \begin{array}{c} n \\ \downarrow \\ 10^{n-1} \end{array}$$

### 3. Discovering a rule

Beginning this lesson, the teacher should do some recapitulation:

> When a sequence has just a few terms, we write them all. When the number of terms is big or the sequence is nonterminating, we use the symbol " ... " which carries the command: continue as it has been done. It is just great when you discover a rule and know to solve the problem:

$$\begin{array}{c} n \\ \downarrow \\ \_\_\,. \end{array}$$

Now the attention is focused on examples where the rule of formation has the form "$an \pm b$". First examples should be slight variations of a), b) (and c)) in 2. Ex. 4.

$$\begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 2 & 3 & 4 & & \_\_, & \end{array} \qquad \begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 3 & 5 & 7 & & \_\_, & \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 4 & 6 & 8 & & \_\_. & \end{array}$$

**1.** Discover a rule and fill in

$$\text{a)} \begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 3 & 6 & 9 & & \_\_ & \end{array} \qquad \text{b)} \begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 2 & 5 & 8 & & \_\_ & \end{array}$$

$$\text{c)} \begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 5 & 8 & 11 & & \_\_ & \end{array} \qquad \text{d)} \begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 5 & 10 & 15 & & \_\_ & \end{array}$$

$$\text{e)} \begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 8 & 13 & 18 & & \_\_ & \end{array} \qquad \text{f)} \begin{array}{cccccc} 1 & 2 & 3 & & n & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 3 & 8 & 13 & & \_\_. & \end{array}$$

Several similar examples can be given. Let us note that main objective here is not wittiness in discovering rules but the skill in composition of expressions which formulate them. That is why the group of examples which do not vary much should be given. When trained enough, children discover by themselves that the difference of consecutive terms is constant. This difference is $a$ in the expression $an + b$ and then, $b$ is easily found. With acquired routine, children solve problems without organized and arranged data. For instance, as the following one.

**2.** Discover a rule for each sequence:

a) 6, 13, 20, ... b) 9, 19, 29, ...

The problems of finding place for given term are also interesting.

**3.**

$$\text{a)} \begin{array}{cccccc} 1 & 2 & 3 & & \_\_ & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 6 & 11 & 16 & & 496 & \end{array} \qquad \text{b)} \begin{array}{cccccc} 1 & 2 & 3 & & \_\_ & \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 5 & 12 & 19 & & 698 & \end{array}$$

For example, solving 3. a), first the rule is found

$$\begin{array}{c} n \\ \downarrow \\ 5 \cdot n + 1 \end{array}$$

and then the equation $5 \cdot n + 1 = 496$ is solved: $n = 99$.

**4.** Find the number of terms of the following sequences

a) 3, 7, 11, ... , 443 b) 7, 13, 19, ... , 2311.

Solution. a)

$$\begin{array}{c} n \\ \downarrow \\ 4 \cdot n - 1 \end{array}, \qquad 4 \cdot n - 1 = 443, \quad n = 111, \quad \text{number of term } 111.$$

Slight variations of the examples

a) 1, 4, 9, ... b) 10, 100, 100, ...

may also be used for discovering a rule.

At the end, we also note that this is an instance in which an informal use of the principle of mathematical induction is seen at work. And inducing hypothesis is one of the most interesting forms of learning by discovery.

## 4. Finding sums

I will describe here a lesson given to a group of gifted fourth graders. They had been acquainted with the content covered in the previous three paragraphs. When I wrote the sum

$$1 + 2 + 3 + \cdots + 100$$

and asked what it was, they quickly answered that it was the sum of first hundred natural numbers. They also suggested the calculation of the sum

$$101 \cdot 50 = 5050.$$

Evidently, they knew the famous story of Gauss. Then, I gave the following two problems

a) $2 + 4 + 6 + \cdots + 100$ b) $1 + 3 + 5 + \cdots + 99$.

"Sums of fifty first even (odd) numbers", said the children and they easily found

a) $102 \cdot 25 = 2550$ b) $100 \cdot 25 = 2500$.

"Very good", I said and I exposed my solutions: b) the same as they did, a) $2500 + 50 = 2550$. They readily explained this my little trick.

As the following problem, they had to find the sum

$$1 + 4 + 7 + \cdots + 3331.$$

I gave the hint: To find the number of summands, consider the sequence

$$1, \quad 4, \quad 7, \quad \dots, \quad 3331.$$

After a while, a girl (Lana) announced her solution

$$(3334 : 2) \cdot 1111 = \cdots = 1\,850\,926$$

(I only had to add the brackets).

Then I changed the sum a little bit, and the problem was to find

$$1 + 4 + 7 + \cdots + 3334.$$

Many hands were raised and the solution was

$$1\,850\,926 + 3334 = 1\,854\,260.$$

"Oh, yes", I said and then, I assigned the problem

$$1 + 4 + 7 + \cdots + 6934.$$

The same girl was first to solve

$$\begin{aligned}(6935 : 2) \cdot 2312 &= 6935 \cdot (2312 : 2)\\ &= 6935 \cdot 1156 = \cdots + 7\,976\,860.\end{aligned}$$

"Bravo", I said but feeling a little bit of anxiety (caused by the fact that $6935 : 2$ is not an integer), I also suggested the solution (which was already spontaneously discovered):

$$(1 + 4 + 7 + \cdots + 6931) + 6934.$$

After a number of similar problems had been done, we turned to solve the following type of problems.

**1.** Given the sequence 3, 8, 13, ... , find the sum of its consecutive terms

$$98 + 103 + \cdots + 998.$$

**2.** Given the sequence 4, 7, 10, ... , find the sum of its consecutive terms starting with 50th and ending with 99th one.

In the case of a few pupils, finding the number of terms was a difficulty and they did it this way $1000 - 100 = 900$ in the former and $99 - 50 = 49$ in the latter problem. To intervene, I turned their attention to the number of first missing terms what they found easier to follow.

At the end, we had such a problem as

**3.** Using "$n$", write the sum of $n$ first terms of the sequence

a) 3, 8, 13, ... b) 4, 7, 10, ...

Children were quick to solve

a) $((5n + 1) : 2) \cdot n$ b) $((3n + 5) : 2) \cdot n$

In order not to care about the parity of, say, $5n+1$, the children were motivated to derive the rule

$$(m:n)\cdot k=(m\cdot k):n$$

and I added that the rule was valid whenever the right-hand division is feasible.

It was our adventure to treat arithmetical progression embodied in concrete subject matter of fourth grade.

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