THE TEACHING OF MATHEMATICS
2001, Vol. IV, 1, pp. 35–40

THE FIRST LECTURE ON NON-CLASSICAL LOGICS

Alexandros Pefku

Abstract. The implication fragment of Heyting's logic is the minimal logical system for which deduction theorem holds. This is known, but not emphasized enough. In this note we present an elementary proof of this fact and propose how to use it in order to promote the concept of many-valued logics in a natural way, by giving a concept for the first lecture on this subject.

AMS Subject Classification: 03B20, 03B55, 03B50, 03F03

Key words and phrases: Non-classical logics, deduction relation, deduction theorem, Heyting's logic.

0. Introduction

The aim of this paper is to present the concept of an introductory lecture to the theory of non-classical logics. The level of this lecture supposes just the knowledge of the elementary properties of traditionally defined deduction relation. The central idea is to introduce a simple system of non-classical logic on the basis of the deduction relation. Due to its simplicity, it seems that this consideration could be of didactical interest.

The fact that the implication fragment of Heyting's (or intuitionistic) logic is the minimal logical system for which Deduction Theorem holds we use as one of the crucial arguments to support the thesis that Heyting's logic appears in a quite natural way. This fact is known (see [1] or [6]), but not emphasized enough. In this paper-lecture we present a simple proof of this fact. For this purpose we introduce a sequent calculus **DT** (Deduction Theorem), obviously minimal system for which deduction theorem holds, and then prove that the implication fragment of Heyting's logic $\mathbf{LJ}_{\rightarrow}$ and **DT** coincide.

1. The deduction (or consequence) relation

We suppose that our *propositional language* consists of (1) a denumerable set of propositional letters: $\{p_1, p_2, \dots\}$, (2) one binary logical connective, the implication symbol: $\rightarrow$ and (3) two auxilliary symbols, the paretheses:) and (. The *set* For *of (propositional) formulae* is the smallest set containing propositional letters and closed under the following formation rule: if A and B are formulae, then $(A \rightarrow B)$ is a formula. Capitals $A, B, C, D, \dots$, with or without subscripts, are metavariables

The work in this paper was supported in the part by Ministry of science and technology of Serbia, grant number 1335.

ranging over the set For. We also suppose that, where there are several occurrences of implication, the first one appearing on the left has the highest priority, i.e., we will use the abbreviation $A \to B \to \cdots \to C \to D$ for $(A \to (B \to (\cdots \to (C \to D)\cdots)))$.

A *logical system* $\mathcal{L}$ is usually defined inductively. The basis of each logical system $\mathcal{L}$ is its *deduction* (or *consequence*) *relation,* denoted by $\vdash_{\mathcal{L}}$ (or, simply, by $\vdash$) and defined as follows: $\Gamma \vdash A$ iff there exists a finite sequence of formulae $A_1, \ldots, A_n$ such that $A_n = A$ and each formula A_i $(1 \leq i \leq n)$ of this sequence satisfies one of the following conditions: *(i)* A_i is an axiom of $\mathcal{L}$, *(ii)* $A_i \in \Gamma$, or *(iii)* A_i is an immediate consequence of a set of formulae $\Pi \subseteq \{A_1, \ldots, A_{i-1}\}$ by an inference rule of $\mathcal{L}$. In such a case we say that from the set of hypotheses Γ in $\mathcal{L}$ we can *infer* A. If $\Gamma = \emptyset$, we say that A is *provable* in $\mathcal{L}$, or A is a *theorem* of $\mathcal{L}$, and denote this fact by $\vdash_{\mathcal{L}} A$. The corresponding finite sequence of formulae terminating by A is a *deduction* of A from Γ in $\mathcal{L}$ or, for $\Gamma = \emptyset$, this will be a *proof* for A in $\mathcal{L}$.

From the above definition it follows immediately that the consequence relation $\vdash \subseteq \mathcal{P}(\mathrm{For}) \times \mathrm{For}$, where $\mathcal{P}(\mathrm{For})$ is the power set of the set For of formulae, has the following basic properties:

(i) If $A \in \Gamma$, then $\Gamma \vdash A$.

(ii) If $\Gamma \vdash A$, then $\Gamma \cup \{B\} \vdash A$.

(iii) If $\Gamma \vdash A$ and $\Pi \cup \{A\} \vdash B$, then $\Gamma \cup \Pi \vdash B$.

For unions $\Gamma \cup \Pi$ and $\Gamma \cup \{A\}$ of the sets of formulae we will use the following denotation Γ, Π and Γ, A, respectively.

In close connection with the deduction relation is th notion of *deductive closure.* Let $\mathcal{L}$ be a logical system. Then, for any set Π of formulae its deductive closure $\mathrm{Cn}(\Pi)$ may be defined as follows

$$\mathrm{Cn}(\Pi) = \{B | \Pi \vdash B\}$$

Deductive closure Cn is a special case of the closure operator, usually defined in the context of general topology, having the following remarkable properties:

(i) $\Gamma \subseteq \mathrm{Cn}(\Gamma)$

(ii) If $\Gamma \subseteq \Pi$, *then* $\mathrm{Cn}(\Gamma) \subseteq \mathrm{Cn}(\Pi)$.

(iii) $\mathrm{Cn}(\mathrm{Cn}(\Gamma)) \subseteq \mathrm{Cn}(\Gamma)$

for any sets Γ and Π of formulae. Let us note that these properties of closure operator corresponds exactly to the properties *(i)–(iii)* of deduction relation.

2. The deduction theorem

In this context, supposing that the set of formulae is built up over the language containing the implication connective $\to$, the Deduction Theorem may be formulated as follows:

Deduction Theorem. *For every set* Γ, A, B *of formulae,*

$$\Gamma, A \vdash B \quad \textit{iff} \quad \Gamma \vdash A \to B.$$

In other words, we can say that the Deduction Theorem holds for a logical system if this system is closed under the following two inference rules:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash A \to B}{\Gamma, A \vdash B}$$

The Deduction Theorem shows that, in some way, the implication defined by the considered calculus agrees with the corresponding deduction relation.

The Deduction Theorem is sometimes called the Tarski-Herbrand theorem because A. Tarski and J. Herbrand were the first logicians who mentioned this statement in their works (see [2], [3], [7] and [8]). Let us mention that S. Jaśkowski as well has used the Deduction Theorem as a kind of inference rule (in his paper [5] written several years before its publication).

A simple way to describe precisely the deduction relation is to express its basic properties by means of a pure implicative calculus. This calculus will be denoted by $\mathbf{H}_{\to}$ ($\mathbf{H}$—for the Heyting calculus (A. Heyting) and $\to$—for its implicative fragment). The *axiom schemata* of $\mathbf{H}_{\to}$ are:

$$(A1) \quad A \to B \to A$$

$$(A2) \quad (A \to B \to C) \to (A \to B) \to A \to C$$

and the only inference rule is *modus ponens*:

$$\frac{A \qquad A \to B}{B}$$

Note that axiom schemata represent infinitely many axioms obtained by all possible choices of formulae A, B and C. Similarly, according to *modus ponens*, from any two formulae of the form A and $A \to B$, we can infer B, as an immediate consequence in $\mathbf{H}_{\to}$.

Implicative fragment $\mathbf{H}_{\to}$ of the Heyting logic is known also as Hilbert's positive implicational calculus (see [4]). Let us remark that an extension of $\mathbf{H}_{\to}$ by the axiom scheme

$$((A \to B) \to A) \to A \qquad \textit{(Peirce's rule)}$$

is a proper extension of $\mathbf{H}_{\to}$ and presents exactely the implicative fragment $\mathbf{C}_{\to}$ of the classical two-valued logic.

By induction on the length of the deduction in $\mathbf{H}_{\to}$ the following well-known statement is provable:

Proposition *The Deduction Theorem holds for* $\mathbf{H}_{\to}$.

Proof. The "only if" part. We use the induction on the length of the derivation of $\Gamma, A \vdash B$ in $\mathbf{H}_{\to}$. Let $A_1, \ldots, A_n = B$ be a deduction of B from Γ, A. We will show that, for each i $(1 \leq i \leq n)$, a deduction of $A \to A_i$ from Γ can be constructed and, consequently, a deduction of $A \to B$. The folowing cases are possible:

(i) A_i is an axiom or $A_i \in \Gamma$. Then from A_i and the axiom $(A1)$ $A_i \to A \to A_i$, by (mp), we infer $A \to A_i$.

(ii) $A_i = A$. Then the identity law $A \to A$ justifies our conclusion.

(iii) For some $j, k < i$, we have $A_k = A_j \to A_i$. By the induction hypothesis, we have deductions of $A \to A_j$ and $A \to A_j \to A_i$ from Γ. Then, from $A \to A_j \to A_i$ and the axiom $(A2)$ $(A \to A_j \to A_i) \to (A \to A_j) \to A \to A_i$, by (mp), we have $(A \to A_j) \to A \to A_i$. From the last formula and $A \to A_j$, finally, by (mp), we infer $A \to A_i$, i.e., for $i = n$, we have $A \to B$.

The "if" part is almost trivial. From any deduction of $\Gamma \vdash A \to B$, by (mp), having A as an additional hypothesis, we infer $\Gamma, A \vdash B$. ■

By inspection of the proof just presented, *a fortiori,* we have the following statement:

CONSEQUENCE. *The Deduction Theorem holds for* $\mathbf{C}_{\to}$.

But it is not clear yet that $\mathbf{H}_{\to}$ is the minimal system for which Deduction Theorem holds.

3. The minimal deduction theorem

Let us introduce two Gentzen style sequent calculi based on implication.

The basic notion of the sequent calculus is an expression of the form $\Gamma \vdash A$ which is called *sequent*, where Γ is any word over the set of formulae, i.e., the finite sequence of formulae (without commas). As metavariables, with or without subscripts, for those finite (possibly empty) sequences of formulae we use Greek capital letters $\Gamma, \Pi, \ldots$ Instead of the empty word we leave a blank.

Let us define a sequent calculus $\mathbf{LJ}_{\to}$. The only *axiom scheme* of $\mathbf{LJ}_{\to}$ is the sequent of the form:

$$A \vdash A$$

where A can be any propositional formula. The *structural infrenece rules* of $\mathbf{LJ}_{\to}$ are:

$$\frac{\Gamma A B \Pi \vdash C}{\Gamma B A \Pi \vdash C} \quad \text{(permutation)}$$

$$\frac{\Gamma A A \vdash B}{\Gamma A \vdash B} \quad \text{(contraction)}$$

$$\frac{\Gamma \vdash B}{\Gamma A \vdash B} \quad \text{(weakening)}$$

$$\frac{\Gamma \vdash A \qquad \Pi A \vdash B}{\Gamma \Pi \vdash B} \quad \text{(cut)}$$

The *logical inference rules* of $\mathbf{LJ}_{\to}$ are:

$$\frac{\Gamma A \vdash B}{\Gamma \vdash A \to B} \quad (\vdash\to)$$

$$\frac{\Gamma \vdash A \qquad \Pi B \vdash C}{\Gamma \Pi A \to B \vdash C} \quad (\to\vdash)$$

Let us note that structural rules present the basic properties of the consequence relation and that the logical rules ($\vdash\rightarrow$) and ($\rightarrow\vdash$) are closely connected with the Deduction Theorem. More accurrately, the axiom, the weakening rule and the cut rule of $\mathbf{LJ}_{\rightarrow}$ corresponds exactly to the basic properties *(i)–(iii)* of deduction relation.

By induction on the length of the deduction in $\mathbf{H}_{\rightarrow}$ and by induction on the length of the proof in $\mathbf{LJ}_{\rightarrow}$ one can prove that the Hilbert type system $\mathbf{H}_{\rightarrow}$ and the Gentzen type system $\mathbf{LJ}_{\rightarrow}$ define the same logic:

THEOREM. *For arbitrary formulae $A_1, \ldots, A_n, A$, the formula $A_1 \rightarrow \cdots \rightarrow A_n \rightarrow A$ is provable in $\mathbf{H}_{\rightarrow}$ iff the sequent $A_1 \ldots A_n \vdash A$ is provable in $\mathbf{LJ}_{\rightarrow}$.*

Now we will introduce the sequent calculus **DT** (Deduction Theorem) which consists of the axiom and structural rules of $\mathbf{LJ}_{\rightarrow}$, the following two **DT**-rules:

$$\frac{\Gamma A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \frac{\Gamma \vdash A \rightarrow B}{\Gamma A \vdash B}$$

Obviously, *the set of formulae* $\{A|$ the sequent $\vdash A$ is provable in $\mathbf{DT}\}$ *presents the minimal logical system for which the Deduction Theorem holds.* This is why the system **DT** is defined as a deductive closure of **DT**-rules only. We will show that this set coincides with the set of formulae provable in $\mathbf{H}_{\rightarrow}$. It suffices to show that **DT** and $\mathbf{LJ}_{\rightarrow}$ coincide.

The following **DT**-deduction tree:

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \dfrac{A \rightarrow B \vdash A \rightarrow B}{AA \rightarrow B \vdash B}}{\Gamma A \rightarrow B \vdash B} \qquad \Pi B \vdash C}{\Gamma A \rightarrow B \Pi \vdash C}$$

shows that the rule ($\rightarrow\vdash$) is derivable in **DT**, meaning that we have:

THEOREM. *For any sequence of formulae ΓA, the sequent $\Gamma \vdash A$ is provable in $\mathbf{LJ}_{\rightarrow}$ iff $\Gamma \vdash A$ is provable in* **DT**.

Consequently, we can conclude the following:

PROPOSITION. *The implication fragment of Heyting's logic $\mathbf{H}_{\rightarrow}$ is the minimal finitely axiomatisabile logical system, closed for the rules modus ponens and substitution, for which Deduction Theorem holds.*

The Deduction Theorem may be considered a technical result which justifies introducing the system $\mathbf{H}_{\rightarrow}$ as a natural formalization of the consequence relation, but the above statement, defining the system $\mathbf{H}_{\rightarrow}$ in a unique, simple and, most of all, quite natural way, must be of the great importance. The system $\mathbf{H}_{\rightarrow}$ is an example of non-classical logical calculus, incomlete with respect to the usual two-valued classical semantics. Note that the Deduction Theorem holds *a fortiori* for each extension of $\mathbf{H}_{\rightarrow}$, over the propositional language, closed for substitution, with *modus ponens* as the only rule of inference.

The next lecture on the same subject should contain the proof that the implication fragment of the Heyting logic is an infinitely-valued logic.

REFERENCES

1. D. M. Gabbay, *Semantical Investigations in Heyting's Intuitionistic Logic,* D. Reidel Publ. Comp., Dordrecht, 1981.
2. J. Herbrand, *Sur la théorie de la démonstration,* Comptes-Rendus hebdomadaires de séances de l'Académie des Sciences de Paris 186 (1928), pp. 1274–1276.
3. J. Herbrand, *Recherches sur la théorie de la démonstration* (thése), Prace Towarzystwa Naukowego Warszawskiego, Wydzial 3, 33 (1931), pp. 35–153.
4. D. Hilbert, P. Bernays, *Grundlagen der Mathematik*, Band II, Springer, Berlin, 1939.
5. S. Jaśkowski, *On the rules of supposition in formal logic,* Studia Logica 1 (1934), pp. 5–32.
6. W. A. Pogorzelski, *On the scope of the classical deduction theorem,* The Journal of Symbolic Logic 33 (1968), pp. 77–81.
7. J. Porte, *Fifty years of deduction theorems,* Proceedings of the Herbrand Symposium, Logic Colloquium '81, (J. Stern, ed.), North-Holland, Amsterdam, 1982, pp. 243–250.
8. A. Tarski, *Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften I,* Monatshefte für Mathematik und Physik 37 (1930), pp. 361–404. (A revised translation appeard as: *Fundamental concepts of the methodology of the deductive sciences,* Logic, Semantics, Metamathematics, Papers from 1923–1938, Clarendon Press, Oxford, 1956, pp. 60–109.)

University of Belgrade, Faculty of Economics
Kamenička 6, 11000 Belgrade, Yugoslavia
E-mail: boricic@one.ekof.bg.ac.yu