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METRIC EUCLIDEAN PROJECTIVE AND TOPOLOGICAL PROPERTIES

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Abstract. This paper is devoted to those readers who are professionally engaged in dealing with the questions of teaching and learning elementary school geometry. It consists of three lessons from a postgraduate course taught by this author at the Teacher's Training Faculty, University of Belgrade.

Appearance of things in the surrounding world changes but some stable characteristic properties of their shapes stay unchanged. J. Piaget classifies these properties as topological projective and Euclidean and the spontaneous development of a child follows that order of ideas. It is a normal interest of a specialist in education to know how these ideas are mathematically established without wading through the books on these subjects which are often unapproachable to him or her. The aim of this paper is to make a direct approach to mathematical clarification of these ideas, based only on the reader's knowledge of the secondary school mathematics.

AMS Subject Classification: 00 A 35

Key words and phrases: Metric properties, Euclidean properties, projective properties, topological properties.

1. A search for lost meanings

Si donc il n'y avait pas de corps solides dans la nature, il n'y aurait pas de géométrie.
H. Poincaré

From the rise of first human beings until the time of Greek schools of philosophy and mathematics, geometrical ideas had been inherent and they sprang as the result of man's efforts to establish a more intelligent relationship with the variety of shapes of physical objects existing in the surrounding space. At the Grecian time, particularly starting with Pythagorean school, such ideas became inner representations of abstract concepts, certainly developed on the basis of visualization but logically used in a way independent of the generating processes.

As we know it from history of mathematics, Thales of Milletus proved that an angle inscribed in a semicircle is a right angle relying his arguments on the other simpler and more evident facts. In a period of about three centuries, from Thales to Euclid, the ancient Greek geometers reduced the number of simple, evident geometrical facts to a system of a priori acceptable, initial truths called postulates, which the reader certainly knows from his or her school geometry course. Acceptance of such a system of axioms and deduction of all other geometrical facts from it, turned the Greek geometry into a perfect, logically organized subject.

It was Euclid who exposed all main contributions of Greek mathematics in his thirteen books of "The Elements". For centuries "The Elements" served as the only source for learning geometry. Even when some simplified versions were written, it was still difficult to approach geometry using them. Only older students with the help of their masters were able to grasp the contests of this subject. With a sequence of basic concepts denoted by undefined terms and those, given by definitions, with axioms as a basis for deductive conclusion, such a course of geometry had to be difficult for a beginner who lacked the necessary imagination, which could be formed only in touch with more concrete realizations of geometrical ideas. Didactically motivated, further and further simplifications of "The Elements" have been made, resulting in what we now see as school books in Euclidean geometry.

Following of the Euclidean tradition has always been stressed as the best way of development of logical thinking. To make possible a successful approach to such abstract geometrical thinking, the necessity of training in visualization has also been felt for a longer time. The models of solid geometric bodies, which could be seen in classrooms were an evident manifestation of these efforts.

Euclid did not write his books to be used by children in schools. "The Elements" were designed as a scientific treatise which have been a model of perfect and rigorous exposition. Therefore nothing bad exists in a course following Euclid, the preparation of students for such a course may be bad.

Different standpoints on what geometry is and how has to be taught are exposed in Freudenthal [4], chapter titled "The case of geometry". Expressing his own view, Freudenthal says: "Geometry can only be meaningful if it exploits the relation of geometry to the experienced space. If the educator shrinks this duty, he throws away an irretrievable chance".

In the same chapter, the work of educators in the Netherlands who approach introductory geometry as science of physical space, is presented. From nineteen twenties onwards Tatiana Ehrenfest-Afanassjewa created her propedeutic geometry using concrete material and letting children be experimenting with it. The van Hieles and van Albada followed this approach to geometry enriching concrete material and expanding activities to include: paper folding, cutting, gluing, drawing, painting, measuring, paving and filling.

Up to which degree such activities could be mere playing for children and do the encountered miracles of space impress them is one thing to question, the other one is indisputable that they contribute much to the bridging of the gap between reality and geometrical abstractions. On the other side, for majority of children, geometry should not be a challenge for their intelligence but a rational relationship with the surrounding reality.

The investigations of the ways how children form geometric concepts, carried out by J. Piaget, have certainly inspired curriculum planners to enrich geometric contents of the first classes of elementary school and to establish an order of ideas which follows spontaneous development of the child. According to the Piaget's experimental findings: "A child's order of development in geometry seems to reverse the order of historical discovery. Scientific geometry began with the Euclidean system concerned with figures, angles and so on, developed in the 17th century the so-called projective geometry (dealing with problems of perspective) and finally came in the 19th century to topology (describing spatial relationships in general qualitative way—for instance, the distinction between open and closed structures, interiority and exteriority, proximity and separation). ... Not until a considerable time after he [a child] has mastered topological relationships does he begin to develop his notions of Euclidean and projective geometry. Then he builds those simultaneously." [7, p. 3].

Geometrical concepts usually scheduled in the curriculum of the first four classes of elementary school begin to exist at the sensory level. Gradually, by means of their iconic representation and through their verbal expression, they become more and more abstract, approaching so the level of abstractness supposed in the Euclidean geometry. The efforts to establish a new more concrete and realistic meaning to these concepts we call here the search for lost meanings.

And to show somewhat clearer and so more technically what metric (Euclidean), projective and topological properties of geometric objects are, we write the following cections, starting first with the description of the structure and function of the eye.

2. How the eye functions

Perception invites comprehension, that is the thought mechanism by which the mind manipulates the perceived. And that what is comprehended may also be expressed in words or symbols. This looks like a road from the outside to the inside of the mind and backwards. The outer part of this road is clearer and now we turn our attention to it.

According to Aristotle, there are five, now called, classical senses: vision, hearing, touch, taste and smell (and contemporary psychology considers some others: movement sense, sense of balance, etc.). The greatest amount of information about the surrounding world is gained through our sense of vision, which also has been studied more than any of the others.

We see objects, which emit or reflect light and, quite obviously, using our eyes. A simplified scheme of the eye is given in Fig. 1, and its functioning can be compared to that of a camera.

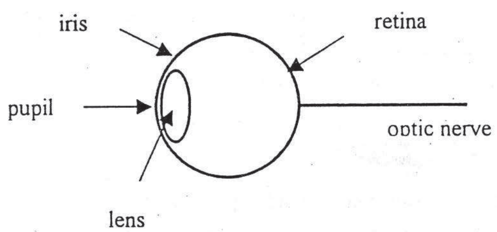

Fig. 1

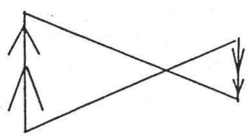

Fig. 2

Pupil is the opening of the eye which admits light (aperture of the camera), lens which focuses light, being a transparent bag filled with a half-fluid, half-solid crystalline substance (camera lenses) and the back wall of the eye, called retina upon which the light image is cast (film of the camera). The retina is capable of continuous operation recording thousands upon thousands of images throughout our walking hours. Iris is the front wall of the eye (its colour is the colour of your eyes) and its muscles, which form pupil are controlled by the amount of light (while the camera aperture is set).

The light projected by an object that we see forms its image on retina, which stands upside down (Fig. 2).

Since the retina is a two-dimensional surface, we see only what is projected upon that surface. How then we see depth, which is the third dimension? There is a number of cues for it, called monocular (one-eye) which allow a single eye to see some depth or binocular (two-eye) which arise in looking with both eyes.

Monocular cues are:

Overlap, when one object blocks part of the view of another object and when the blocked object appears to be farther away.

Linear perspective, when the farther away an object is, the smaller its image on the retina is.

Haziness, when distant objects appear hazy.

Shadows, when the farther away an object is, the more shadowy is its view.

As a binocular cue, retina disparity is of highest interest. Since each of our eyes views an object from a slightly different angle, two its images on two retinas are slightly different. The famous painter Leonardo da Vinci was the first to diagram this effect (Fig. 3).

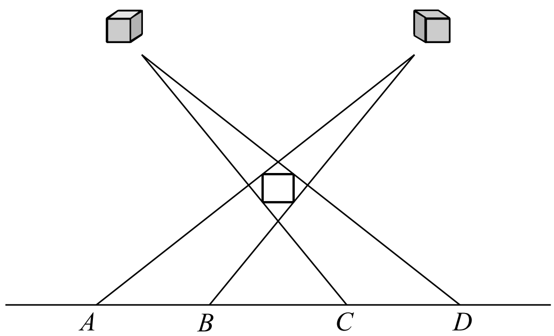

Fig. 3

Being blocked by a cube, the left eye does not see the part AB and the right eye the part CD of the presented line. They both see the whole line and the shaded area is seen by neither of them. Together the two eyes can see around the cube taking in almost the whole background, what causes the effect of depth. The two cubes in Fig. 3 look different, showing so what each of the eyes sees.

3. Metric and Euclidean properties

One of the man's fundamental ideas is that of equal things. The way, how to compare things and create patterns to produce them have been occupying the man's mind at all times. For instance, today, the mass production of necessities of life depends upon our ability to make parts that are almost exactly alike, what is to say equal in size and shape.

Attempting to exhibit the unity of perception and geometry, first we express natural dependence of geometric ideas on the features of solid bodies in the surroundings. Namely, a solid body A is conceived as a pure geometrical idea $\overline{A}$, when we forget all of its physical properties. Then, the idea $\overline{A}$, taken to exist independently of A, is called a geometric object. To be more precise, two solid bodies A_1 and A_2 can be different and the two ideas $\overline{A}_1$ and $\overline{A}_2$ equal. For instance, a wooden ball A_1 and a plastic one A_2, both of diameter 10 cm are examples of different solid bodies, producing equal geometric ideas $\overline{A}_1$ and $\overline{A}_2$, respectively.

In [4], Chapter XVI, Freudenthal quotes a case from Diana van Hiele's lesson on congruence. The teacher's first examples were the congruent chairs in the classroom. The way how the students expressed congruence was very beautiful indeed: "The objects that cannot be distinguished." But should we talk of congruent chairs or of them being equal in size and shape is a matter to think of.

Two physical objects equal in size and shape produce two images in mind, which differ by their position in the inner space, and if we have to conceive them as pure geometric objects, only their positions make them different. Representing them iconically, two drawings on a leaf of paper should be equal in size and shape and then, we speak of congruent figures represented by iconic signs.

Space is a fundamental category, basic to all our cognitive processes. The French philosopher H. Bergson says: " . . . the higher we rise in the scale of intelligent beings, the more clearly do we meet with the independent idea of a homogenous space." [2].

In mathematics, a straight line is taken to be a one-dimensional space. When a point O is fixed on such a line, then a one-to-one correspondence between the set of points of the line and the set of real numbers can be established.

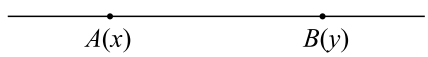

Fig. 4

Two points A and B with the attached numbers x and y are at the distance which is measured by the length of the segment AB, what will be the number

$$d(A, B) = |x - y|.$$

A plane is a two-dimensional space. When it is supplied by a coordinate system, then a one-to-one correspondence between the points of that plane and the set of all ordered pairs of real numbers can be established.

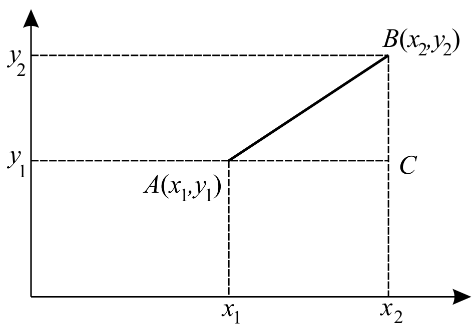

Fig. 5

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the length of the straight line segment AB, what is the number

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

When we speak of three-dimensional space, we think of outer, physical space in which everything real exists. But it is wrong to situate our abstract geometric ideas in it.

When we imagine a thing, we necessarily "see" it expanding in, what is called, the inner space. Thus, the inner space can be taken as our mental representation of the outer space.

In the Greek geometry, there was no explicit idea of space. But implicitly, the idea of space existed as an epiphenomenon, a frame within which geometric objects expand. In modern geometry, we have its symbolic codification as the set of all ordered triples of real numbers. Observation of outer space as a receptacle of real things and evocation of mental space as a receptacle of imagined things are not the learnt abilities but the natural gifts of human beings. Thus, we never seek for an explanation what the space is, neither we try to find one for our students.

In mathematics, a set of points, which are in one-to-one correspondence with all ordered triples of real numbers is called a three-dimensional space. The three numbers x, y, z of the triple (x, y, z) corresponded to a point A are called its coordinates. The subsets of the space corresponded to these subsets

$$\{\, (x, 0, 0) : x \in \mathbf{R} \,\}, \quad \{\, (0, y, 0) : y \in \mathbf{R} \,\}, \quad \{\, (0, 0, z) : z \in \mathbf{R} \,\}$$

of the set of triples, are called x-axis, y-axis and z-axis, respectively. For two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the straight line segment AB is defined to be the set of points corresponded to the following subset of triples

$$\{\, (\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2, \lambda z_1 + (1 - \lambda)z_2) : \lambda \in [0, 1] \,\}.$$

Then, the length of the segment, and thus the distance of the points A and B, is defined to be the number

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We leave out further details, which are known to the reader from his or her analytical geometry course and which have normally been learned by means of visualization. A drawing representing a coordinate system—three straight lines standing perpendicularly to each other, is what such analytical geometry lessons begin with, while three mutually perpendicular edges of walls of a room are a very good materialization of such a system. Whereas here, in order to attain a distinction

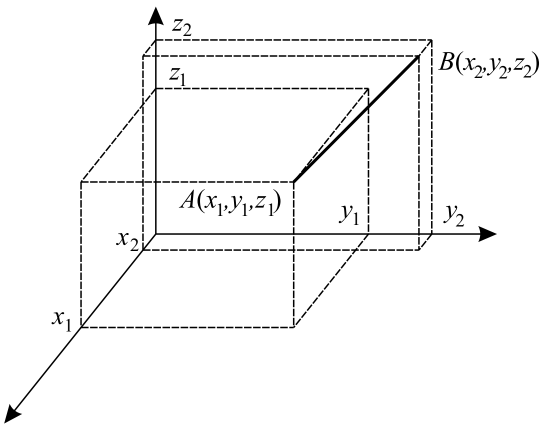

Fig. 6

between the intuitive and the formal, we shortly went into a more logically founded exposition.

Thus, mathematically defined three-dimensional space is denoted by $\mathbf{R}^3$ and is called the Euclidean 3-dimensional space. Then, 1-dimensional space can be identified with x-axis and 2-dimensional space with xy-plane in $\mathbf{R}^3$. These two spaces are denoted by $\mathbf{R}^1$—the Euclidean 1-dimensional space and by $\mathbf{R}^2$—the Euclidean 2-dimensional space, respectively.

Once we have the concept of Euclidean space $\mathbf{R}^3$, we can define a geometric object to be any subset of $\mathbf{R}^3$, though of real interest are only very regular subset as straight lines, circles, triangles, spheres, pyramids, etc.

Let X and Y be subsets of $\mathbf{R}^3$. A mapping $F\colon X \to Y$ is called an isometry if for each pairs of points A and B in X,

$$d(f(A), f(B)) = d(A, B).$$

Thus, we see that isometries are those mappings, which preserve the lengths of straight line segments and that two different points A and B have different images $f(A)$ and $f(B)$. Hence, an isometry $f\colon X \to Y$ is a one-to-one mapping. When such a mapping is also onto, two objects X and Y are said to be isometric or congruent.

Two congruent geometric objects share exactly the same set of properties and they "cannot be distinguished" except by their position in the space, what is not taken to be a relevant geometric property. A property preserved under isometries is called a metric property. For instance, such properties are: lengths, areas, volumes, diameters, measures of angles, etc. The reader certainly knows for many other metric properties from his or her school geometry course, as well as, for the statements, with combinations of equal parts that make triangles congruent.

To prove that two geometric objects X and Y are congruent, we have to find an isometry $f\colon X \to Y$. To prove that X and Y are not congruent, we have to

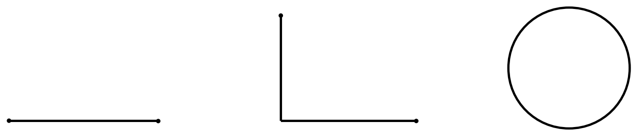

Fig. 7

find a metric property possessed by one of this objects and not by the other. As an example, take the following three lines (Fig. 7)

and, after a look at, we are ready to say that they are not congruent. To prove it, we have to employ some metric properties. Let us say, the following two:

a) For each two different points A and B in X, the segment AB is contained in X.
b) There exists a pair of different points A and B in X, such that the segment AB is contained in X.

The segment in Fig. 7 possesses both properties a) and b), the "el" line only b) and the circle neither of them. Hence these lines are not congruent one to the other.

For the sake of simplicity, we continue to consider the congruence, confining the considerations to the plane $\mathbf{R}^2$. First of all, we will exhibit three important examples of isometries.

1. Translation. Given a vector $\boldsymbol{a}$, the mapping $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, which maps each point $A \in \mathbf{R}^2$ onto the point A' such that the vector $\overrightarrow{AA'} = \boldsymbol{a}$ is called the translation for the vector $\boldsymbol{a}$. (The segments AA' and BB' are parallel and have equal lengths, whence the equality of lengths of AB and $A'B'$ follows.)

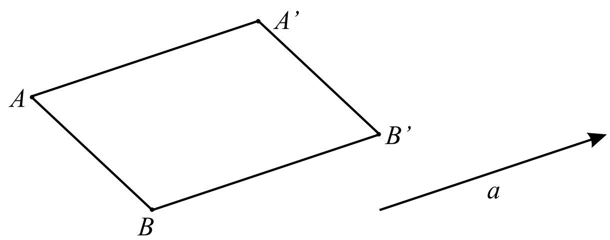

Fig. 8

2. Rotation. Given an angle α and a point $O \in \mathbf{R}^2$ the mapping $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, which maps each point $A \in \mathbf{R}^2$ onto the point A' such that the two segments OA and OA' are equal in length, and form the angle AOA' equal to α, is called the rotation around the point O, for the angle α. ($\angle AOB + \angle BOA' = \alpha$, $\angle BOA' + \angle A'OB' = \alpha$, whence $\angle AOB = \angle A'OB'$. From the congruence of the triangles AOB and $A'OB'$, it follows that the lengths of AB and $A'B'$ are equal.)

3. Symmetry. Given a straight line p in $\mathbf{R}^2$ the mapping $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ which maps each point $A \in \mathbf{R}^2$ onto the point A' such that p is the perpendicular bisector of the segment AA', is called the symmetry with respect to p (and p is called the axis of symmetry). (The two right triangles BCD and $B'CD$ are easily seen to be

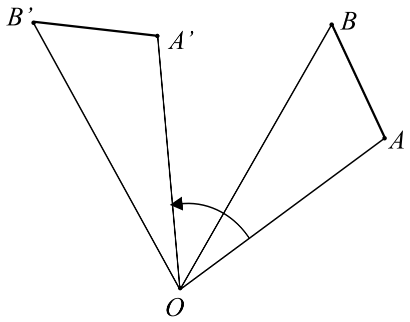

Fig. 9

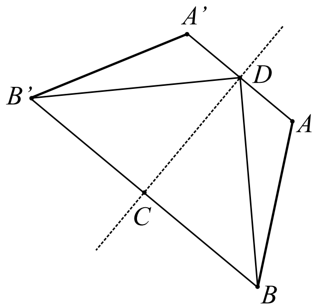

Fig. 10

congruent, what implies $\angle BDA = \angle B'DA'$ and the equality of lengths of BD and $B'D$. From the congruence of triangles BDA and $B'DA'$, the equality of lengths of AB and $A'B'$ follows.)

Let $f\colon X \to Y$ be an isometry. If A, B and C are three non-colinear points in X and A', B' and C' their f-images in Y, then for each point D (belonging to X or not), there exists a unique point D' such that D' is at the same distances from A', B' and C' as the point D is from A, B and C, respectively. Thus, the mapping f is completely determined as soon as we know the images of three non-colinear points and can also be taken as an isometry from $\mathbf{R}^2$ to $\mathbf{R}^2$.

Figuratively speaking, two plane geometric objects are congruent if it is possible to move one of them until it coincides with the other one. The following beautiful theorem says that it can be done in at most two elegant moves.

Each isometry $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ is one of the following mappings:

(I) a translation,

(II) a rotation,

(III) a symmetry,

(IV) the composition of a translation and a rotation,

(V) the composition of a translation and a symmetry.

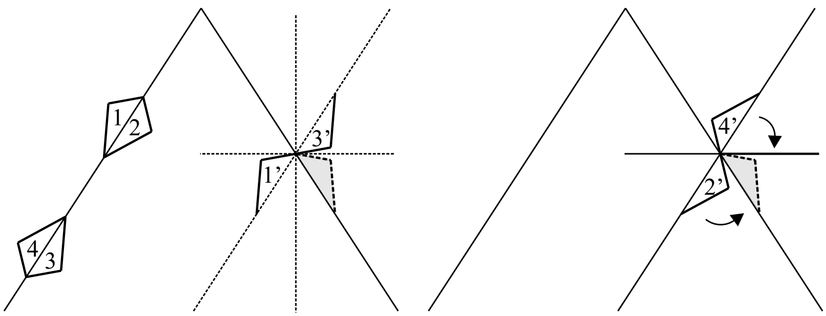

Fig. 11

Relying on the fact that an isometry is completely determined knowing how three non-colinear points are mapped, it is enough to show that a triangle ABC can be made to coincide with its congruent f-image $A'B'C'$ in at most two "elegant moves". The reader will enjoy seeing it by himself or herself, looking at the pictures in Fig. 11. (Translated triangles $1'$ and $3'$ coincide with the shaded one by symmetries with respect to the drawn axes and those $2'$ and $4'$ after two evident rotations. The case, when analogous sides of two triangles are parallel, is treated almost the same way.)

The above theorem is a foundation to define pairs of congruent objects having the same or opposite orientation. If two plane objects are congruent and the congruence can be realized using a mapping under (I), (II) or (IV), then we say that they also have the same orientation. The three such pairs are given in Fig. 12.

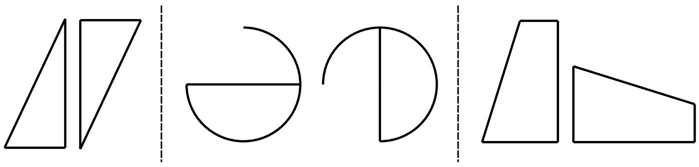

Fig. 12

If the congruence is inevitably realized using (III) or (V), then we say that they have opposite orientations. The three pairs of oppositely oriented objects are given in Fig. 13.

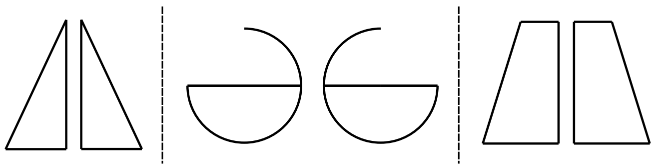

Fig. 13

Figuratively speaking, two congruent figures of different orientation can not coincide moving them in the plane without one of them is turned over. In mathematics, two different orientations are designed using terms "positive" and "negative". In case of real objects whose shapes are of opposite orientation more common terms are "left" and "right". Pairs of your hands, arms, legs, feet, ears, etc. are, when conceived geometrically, the examples of congruent objects having opposite orientations. Things as pairs of shoes, gloves, skis, etc. are such further examples, as well as, left and right keys, left and right doors, etc.

It is very good when children are trained to distinguish two orientations as, for example, the "right" (correct) and the "left" (incorrect) figure six in Fig. 13. It is equally bad when we hear many a teacher saying for two differently oriented figures that they are different in shape. Since there is no excuse for mistakes made at any level of instructing, we now turn our attention to the concept of shape. But first, let us consider one more mapping of the plane onto itself.

4. Homotety. Given a point O in the plane and a positive real number k, the mapping $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, which maps each point $A \in \mathbf{R}^2$ onto the point A' belonging to the ray OA and being such that the ratio of the lengths of the segments OA' and OA is equal to k, is called homotety with the centre O and the coefficient k.

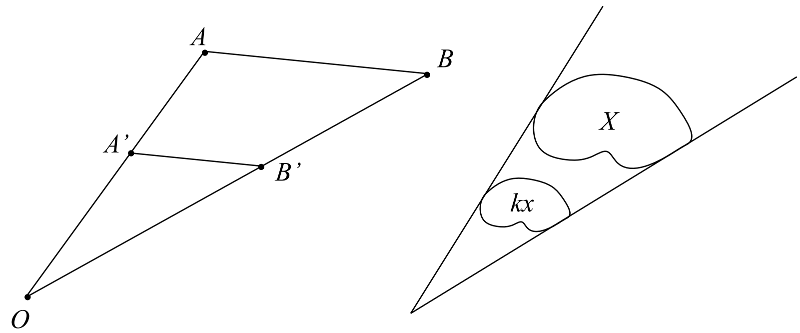

Fig. 14

(From the similarity of the triangles OAB and $OA'B'$, it follows that $d(A', B') = kd(A, B)$.)

For an object $X \subset \mathbf{R}^2$, its homotetic image is denoted by kX and kX can be viewed as, in all directions proportionally shortened (when $k < 1$) or prolonged (when $k > 1$), object X.

In a very similar way homotety is defined in $\mathbf{R}^3$, kX denotes again the homotetic image of X.

Let X and Y be two geometric objects. If there exists a k, $(k > 0)$ such that kX is congruent to Y (or $\frac{1}{k}Y$ congruent to X), then it is said that X and Y are similar or that they have the same Euclidean shape.

A property, which all mutually similar objects possess is called a Euclidean property. In other words, a property is Euclidean if preserved under homoteties. Apart from some very regular geometric objects, generally it is difficult to determine characterizing properties of the shape of an object. Thus, its shape can be considered as the totality of all its Euclidean properties. And as we know it very well, children start geometry in a holistic way, recognizing shapes of objects, before they are taught to separate their properties and analyse them.

Examples of objects having the same Euclidean shape, in other words being similar, are triangles with equal angles, all squares, circles, cubes, spheres, etc.

The ratio of all linear elements of two similar objects is the same number k, the ratio of corresponding areas k^2 and of volumes k^3.

At the end, let us say it that Euclidean geometry studies both metric and Euclidean properties, as we all know it from our school lessons in geometry.

4. Projective properties

Appearance of things changes but some of their stable, characteristic properties stay unchanged. Thanks to them, we can distinguish sorts of things and build our visual concepts, among which, the geometric ones are fundamental. And as the vision lays a ground, the mind produces a shaped setting of the seen things.

When looking at a flat object (page of a book, picture on a well etc.), we always try to adjust our sight keeping the head in a position when the surface on which the object stands and the surface of retina are approximately parallel. Then, the copy of the object projected on retina and the object itself, when they are conceived geometrically, are the examples of two similar figures, as the following drawing suggests it.

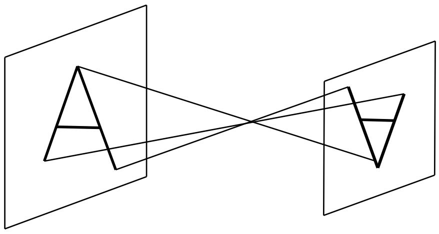

Fig. 15

In such a situation, the objects are seen with full distinction and this way of seeing corresponds to the recognition of all Euclidean properties.

Many objects stand in a position when we cannot have their view in the above way, and when the two planes intersect at some angle. Depending on this angle, the retinal image changes very much. On the other side, viewing a solid body, we see parts of its surface as being projected on retina at different angles. For instance, when an object of the shape of cube is seen and when one of the three visible faces projects as a square, the other two seem as they had oblique angles.

A two-dimensional picture of a three-dimensional real world object can be faithful enough if it produces an image on retina approximately equal to the retinal image of the object itself. But how to make such a picture was certainly a problem encountered whenever such an attempt of presentation was tried. We will sketch here the Christian tradition of decorating church walls by pictorial presentation of the scenes from the Bible, which primarily had the function to instruct the worshippers in the questions of faith.

Developed in antiquity, mosaic is a form of art, in which pictures are made with small pieces of coloured stone and glass. Byzantine churches had their interiors richly decorated with colourful designs and mosaics. The most famous of them all is St. Sophia built in Costantinople, during the reign of Justinian (A.D. 532–565). The cathedral was planed to be flooded with sunlight from a great dome some fifty metres above the floor. The dome rests on four arches, supported by four piers and this impressive construction of stone is still standing today after hundreds of earthquakes, being older than any other great building in Christendom. Thousands of square metres of gold-leaf mosaic on the vaulted ceilings of St. Sophia opened Heaven above the heads of worshipers. Though not on such a large scale, following the model of St. Sophia many churches were built all over the world where

Christianity was spread.

Gilt backgrounds of mosaics suggest that the people and objects exist in some heavenly regions and all figures are symbolic rather than realistic. This kind of art with an evident respect for supernatural has a fascinating beauty, while two-dimensionality is its prominent feature.

Beginning with the tenth century, the fresco painting started to replace mosaics in decorating of the indoor walls of churches. Painting in fresco was, first of all, less expensive but, it also was less hieratic, admitting such devices as light and shade in order to produce a feeling of depth. An especially interesting device in fresco painting was the representing of cubic objects in counter perspective, when the edges of such objects that are farther away look bigger than those that are nearer the front.

The idea of counter perspective is illustrated in Fig. 16, and we direct the interested reader at the book [6], where the 12th century frescos from St. Climent Church (Ohrid, Macedonia) can be seen with the evident presence of counter perspective.

Fig. 16

In the 13th and 14th century, the interest for the Greek and Roman culture revived. Instead of exclusiveness of the plebeian Christianity, a coexistence of faith and science started to be two parallel ways of the search for truth, what the Angelic Doctor, St. Thomas Aquinus (1225–1274) expresses saying that the truth of faith is supra non contra rationem. Thus, the portal of the Renaissance was open, when the art gave evidence of an absorption of secular values, unifying the divine and earthly aspects of existence. Instead in ecstasy, saints were painted in an attitude of quite concentration, ascetic figures of worshippers were replaced by vital, energetic bodies, as well as, convincing presentations of familiar settings of home and landscape were painted.

Obliged to learn mathematics, physics, architecture, stonecutting, metalwork, woodwork, statics, etc., the Renaissance artists were universal men, able to create great paintings, design fortifications, bridges, palaces, churches, etc. This caused the elevation of art from the lower, craft status to that of liberal and theoretical arts. The men as Leonardo, Michelangelo and Raphael achieved great recognition for their profession and the idea of an artist as a man of learning was taking shape.

Linear perspective as a cue for three-dimensional vision was thoroughly studied by the Renaissance artists and used as a new, mighty technique of painting. The painter who set forth the mathematical principles of perspective was Piero della Francesca (1410–1492), who also was the best geometer of his time, considering Euclidean shapes as the purest forms of beauty. His famous painting "The Flagellation of Christ" is seen in Fig. 17 and one should compare it with a "flat" fresco painting to recognize all effects of this new technique of space shaping. (Taken from `http://www.kfki.hu/~arthp/html/p/piero/francesc/flagella.html`)

Leonardo and other Renaissance artists used a perspective grid as the basis

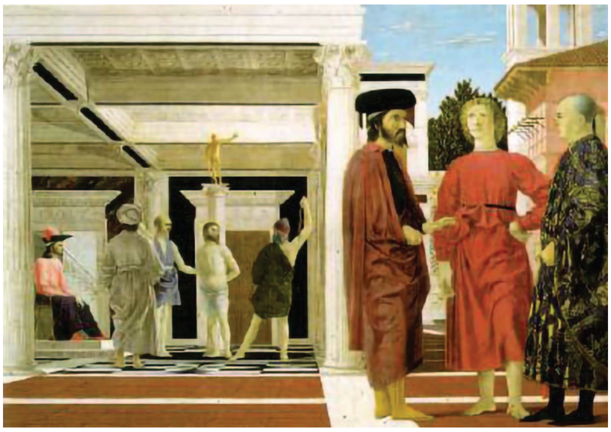

Fig. 17

of space establishing of an entire painting. The theory of perspective, though it lacked a solid mathematical basis, was taught in painting schools together with other liberal arts.

The German painter, Albrecht Dürer (1471–1528), aiming to pass on to his compatriots the knowledge acquired in Italy, innovated the methods of two-dimensional representation of objects, introducing orthogonal projections of curves and human figures on two or three mutually perpendicular planes. This idea was fully developed by the French mathematician Gaspard Monge (1746–1818), who considered geometry as the truth about space and the real world. Finally with mathematical works of the French mathematician Jean-Victor Poncelet (1788–1867), the projective geometry ultimately took shape of a new mathematical discipline.

The reader interested in details related to the history of projective geometry is referred to the voluminous book on history of mathematics by M. Kline ([5]).

In general, as it has already been said, our vision corresponds to the projecting from one plane to another. Now, we give to it a precise mathematical shaping.

Given two planes π and π' and a point P belonging to neither of them (See Fig. 18), for each point $A \in \pi$, the straight line AP intersects π' at the point A'. Thus, a correspondence between the set of points of π and set of points of π' is established and called the *projection from the point* P of the plane π upon the plane π'. Then, it is said that the point A projects upon the point A' or that A' is a projection of A. For a subset X of π, the projections of all its points form a subset X' of π' and X' is also called the projection of X or it is said that X projects upon X'.

In the sunlight you see the trunks of trees projecting parallel shadows. In the light of a street lamp, if the shadows of the trunks were prolonged, they would intersect at the base of the post that holds lamp. This scene from our surroundings show clearly how straight lines are projected.

Now we turn back to geometry and first, we consider the projection of a straight

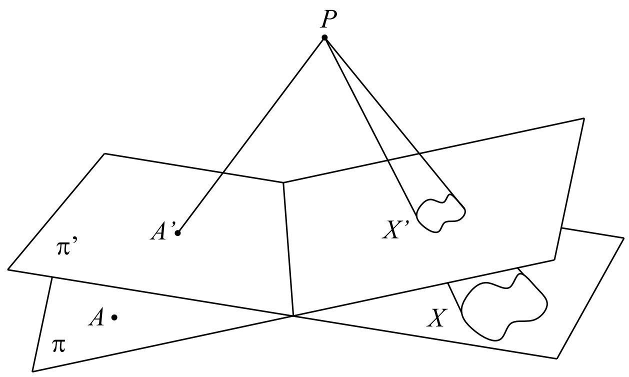

Fig. 18

line. Let p be a straight line belonging to the plane π. All lines passing through a point of p and the point P belong to the plane β determined by the line p and the point P. Hence, all points of p are projected upon the points of a line p' which is the intersection of the planes β and π'.

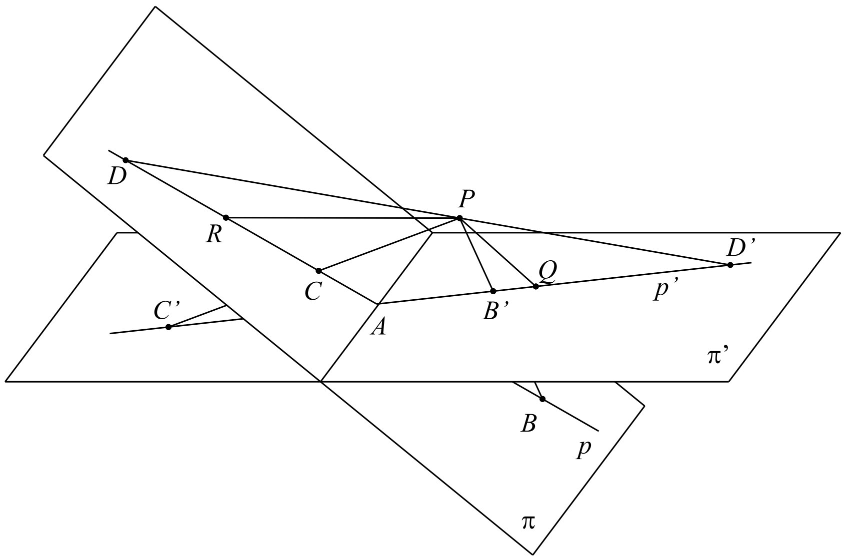

Fig. 19

Let the points Q and R be the intersections of the planes π' and π with the lines through P which are parallel to π and π', respectively. In Fig. 19, a pointwise projection of the line p upon the line p' is seen. Namely, excluding the end points, the ray RD of the line p is projected upon the ray QD' of the line p', the segment AR upon the ray AC' and the ray AB upon the segment AQ. The point A projects upon itself and there is none point of p which would be projected upon Q neither there is a point of p' upon which R would be projected.

To eliminate this deficiency, for each straight line p, its *point at infinity* ∞_p is introduced. With such points, ∞_p projects upon Q and R upon $\infty_{p'}$.

A straight line together with its point at infinity is called the projective straight line and the projection of one such line upon another is a one-to-one and onto mapping of their points. To make a distinction, a line p together with its point at infinity will be denoted by p^*.

The projections of a pair of parallel lines p and q, (Fig. 20), are two, at the point Q, intersecting lines p'^* and q'^*. Then, both points at infinity ∞_p and ∞_q are projected upon the same point Q. It is the reason why these to points are identified and considered to be one and the same point at infinity. Therefore, all mutually parallel lines have the same point at infinity and to two different systems of such lines, two different points at infinity are attached. A plane π together with all points at infinity is called projective plane and we will denote it by π^*.

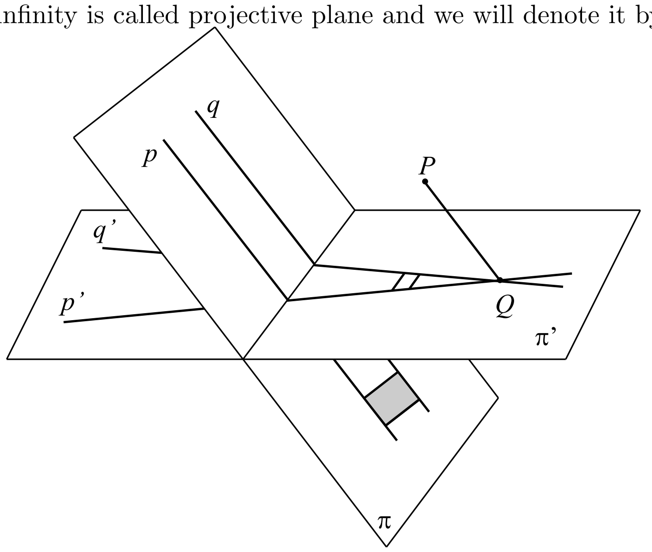

Fig. 20

The projecting of one projective plane upon another is a one-to-one and onto mapping of their points and as such, the mapping has its inverse. This important property of projecting is conditioned by the introduction of points at infinity and the way they are identified. Thus, a projection from a point of π^* upon π'^*, taken as a mapping has for its inverse, the projection from the same point of π'^* upon π^* and, whenever $X \subset \pi^*$ projects upon $X' \subset \pi'^*$, symmetrically considered, $X \subset \pi'^*$ projects upon $X \subset \pi^*$.

In order to avoid a possible confusion caused by the use of analogous terms, a line p and a plane π, taken with their usual meaning, will be called a Euclidean line and a Euclidean plane, respectively, while with the points at infinity attached, p^* and π^* will be called a projective line and a projective plane, respectively. Let us remark that if p and q are parallel Euclidean lines, their corresponding projective lines p^* and q^* intersect at the point at infinity (as $\infty_p = \infty_q$). Hence, each two projective lines intersect and so, there is no idea of parallelism in projective geometry.

To complete this exposition, we will also consider the parallel projecting. Let

π and π' be two planes and p a straight line neither parallel to π nor to π'. Then, p determines a system of parallel lines in the space each of which intersect π at a point A and π' at a point A'. This correspondence of points of the two planes is called the *parallel projection* along the line p. Since all lines of the system determined by p intersect at the same point ∞_p, this correspondence is also called the projection from the point ∞_p of the plane π upon the plane π'.

Now we are in a position to say that a property of planar object X, preserved after an arbitrary number of projecting is called the *projective property.*

In Fig. 20 we see (shaded areas) how a rectangle projects upon a quadrilateral having oblique angles and how a pair of parallel lines projects upon a pair of intersecting lines. The Euclidean properties of a pair of lines to be parallel, of figures to be a square, a rectangle, a parallelogram etc., generally the measures of angles are not preserved under projections and therefore they are not projective properties.

Now we will list a number of projective properties. As the projections of a straight line are again straight lines, we see that the property to be a straight line is a projective property. When three or more points belong to the same straight line, then they are called collinear. The property of three or more points to be collinear is also a projective property. When three or more straight lines intersect at the same point they are called concurrent. The property of three or more lines to be concurrent is again a projective property.

If a line l is curved, its projection l' will also be curved. Indeed, if l' were straight then, projecting it back upon l, the line l would be straight, which is contrary to the assumption. Thus, the property of a line to be curved is a projective property. The straight and the curved are fundamental visual concepts. Preschool children distinguish easily material objects according to these properties as well as the school children distinguish easier straight lines from the curved ones, than they make distinctions on the basis of Euclidean or metric properties.

The property of a line to be zigzag, of a figure to be a triangle, a quadrilateral, etc. are also projective properties.

Geometrically, position of three objects is represented as the configuration consisting of three points. When the objects are aligned, these three points are collinear. Then, one of the objects or one of the points is said to be between the other two. We see therefore that the triple of collinear points and the meaning of the preposition "between" are logically related. Since we often see several types of exercises designed for children to instruct them to use this preposition correctly, we will pay some extra attention to this spatial relationship.

Let us consider two triples of collinear points A, B, C and A', B', C'. We will show how, after two projections, the triple A, B, C can be mapped upon the triple A', B', C'.

Let p be the straight line containing the points A, B, C and q the line containing A', B', C'. Draw the line r through the point A', parallel to p (Fig. 21). First, the triple A, B, C is parallelly projected along the straight line AA' upon the triple A', B_1, C_1. Let P be the point of intersection of the lines $B'B_1$ and $C'C_1$. Then, by projection from P, the triple A', B_1, C_1 maps upon the triple A', B', C'.

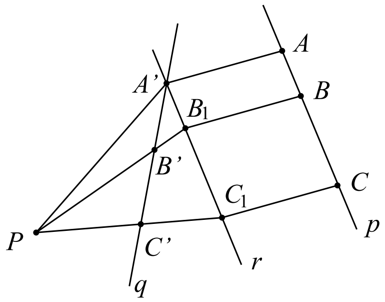

Fig. 21

Since two arbitrary triples of collinear points can be mapped one upon another, we see that under projections neither the distances of points nor the ratios of lengths of segments are preserved. Thus, we see that a metric (distances of points) and a Euclidean (ratios of lengths) property are not projective properties.

But the relationship of points of a collinear triple that one of them is between the other two stays unchanged under projections and so this relationship is a projective property of collinear triples.

Finally, let us observe that a property, which is preserved under all projections, is, of course, preserved under those of them when two planes are parallel. This implies that each projective property is also Euclidean (and metric), but, as it has been shown by several examples, the converse is not true.

At the end, as a statement typical for projective geometry, we formulate the Desargue theorem: Let ABC and $A'B'C'$ be two triangles. If the straight lines AA', BB' and CC' are concurrent, the intersections of pairs of lines AB, $A'B'$; AC, $A'C'$ and BC, $B'C'$ are collinear points, (see Fig. 22).

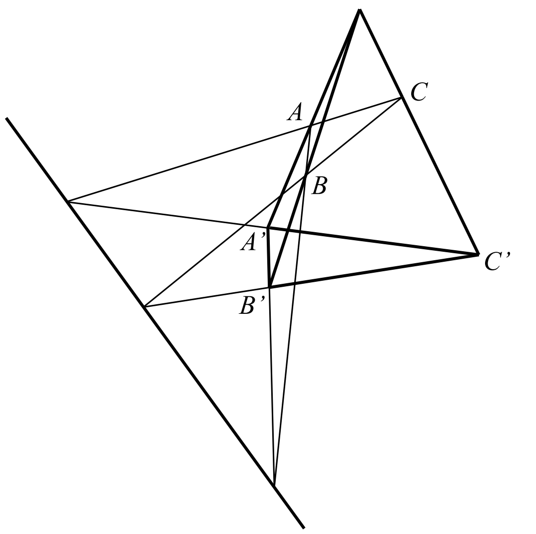

Fig. 22

When the vertices of the two triangles belong to the edges of a trihedral, the intersections of the pairs of lines belong to the line of intersection of the planes determined by surfaces of the two triangles, as it is easy to see. But when the two triangles belong to the same plane, the proof is somewhat more subtle.

5. Topological properties

My contention is that the cognitive operations called thinking are not privilege of mental processes above and beyond perception but the essential ingredients of perception itself.

R. Arnheim

In a purely descriptive way, topology can be described as geometry of plastic deformations. When a geometric object is imagined as being made of ideally plastic substance and when, it is allowed to make it longer, wider—stretching it in all directions or to make it smaller—contracting it in all directions, then the object is said to be under a plastic deformation. More generally, a part of the object can be stretched and, in the same time, the other one contracted. To have this description complete, let us also say that it is not allowed to pull the object into pieces or to let its parts overlap each other. It is not allowed to make a hole in the object, either.

When, as a result of plastic deformation, from an object another one is made, then such two objects are said to be topologically equivalent. In addition, a property preserved under plastic deformations is called *topological property.* Thus, two topologically equivalent objects share the same set of topological properties.

Now we give a number of examples of topologically equivalent objects.

1. Let $[0,1] = \{\, x : 0 < x < 1 \,\}$ be the unit interval. Starting with this interval and deforming it in different ways, the following sequence of lines can be obtained:

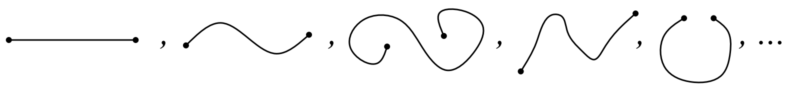

Fig. 23

Each two of these lines are topologically equivalent and one can be obtained from the other by means of a plastic deformation. Each of them represents one and the same thing—a *topological arc*. Among them, the interval has the “nicest” geometric shape, what, by the way, is not a relevant topological property, as we see how the shape of an object can be badly distorted by a plastic deformation.

2. Let $S^1 = \{\, (x,y) : x^2 + y^2 = 1 \,\}$ be the unit circle. Deforming it, a sequence of different lines can be obtained:

Fig. 24

They all are topologically equivalent, representing one and the same thing—a *topological circle*. Geometric circle and equilateral triangle are the "nicest" shapes among them.

3. Let $D = \{ (x, y) : x^2 + y^2 \leqslant 1 \}$ be the unit disk. Deforming it in different ways, a sequence of surfaces is obtained:

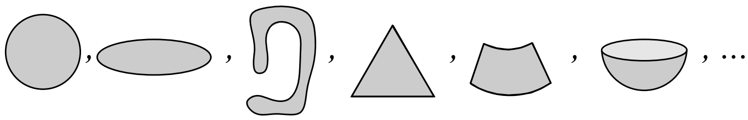

Fig. 25

Each of these surfaces represents topologically one and the same thing—a *topological disk*.

4. Let $C = \{ (x, y, z) : x^2 + y^2 = 1 \text{ and } 0 \leqslant z \leqslant 1 \}$ be the (hollow) cylinder. The lateral surface of a truncated cone, a ring, a rectangle having a circular or square hole, etc. are the objects which can be obtained from the cylinder by the suitable deformations.

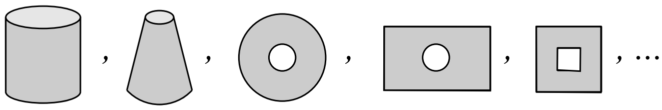

Fig. 26

5. Let $S^2 = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \}$ be the unit sphere. By its different deformations, the objects as the surfaces of a cube, a pyramid, a cylinder etc. are obtained.

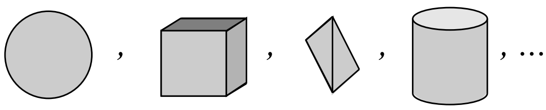

Fig. 27

Formally, two geometric objects A and B are topologically equivalent if there exists a mapping $f : A \to B$, which satisfies the conditions

(1) f is 1–1 and onto,

(2) both, f and f^{-1} (the inverse of f) are continuous.

Such a mapping f is called *homeomorphism* and the two objects A and B are also called *homeomerphic*, what is denoted writing $A \approx B$. It is easy to see that "$\approx$" is an equivalence relation on the set of geometric objects. (f^{-1} is also a homeomorphism as well as it is the composition $g \circ f$ of two homeomorphisms f and g.)

Now we give a number of examples of homeomorphisms.

6. Let (a, b) and (c, d) be any two open intervals. The mapping $f\colon (a, b) \to (c, d)$ given by

$$f(x) = c + \frac{d-c}{b-a}(x-a)$$

is a homeomorphism (Fig. 28, a). Thus, any two open intervals are homeomorphic (topologically equivalent).

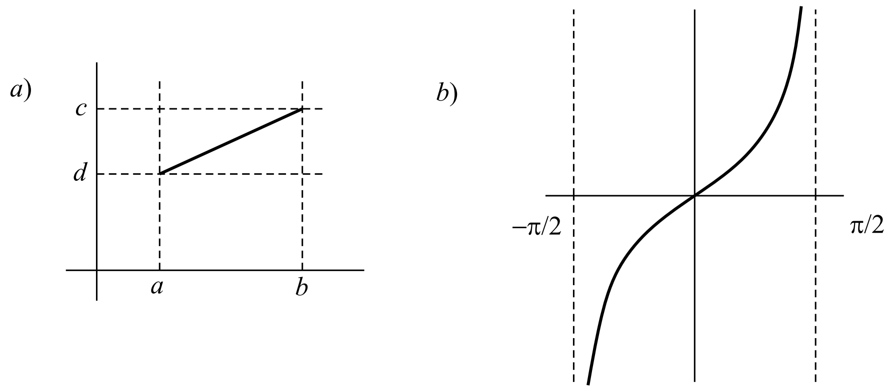

Fig. 28

The mapping $f\colon (-\pi/2, \pi/2) \to \mathbf{R}$, given by $f(x) = \tan x$ is a homeomorphism ($f^{-1}(x) = \arctan x$ is continuous). Thus, we see that the real line $\mathbf{R}$ is homeomorphic to any open interval.

7. The mapping

$$f(t) = \begin{cases} (1-2t, 0), & t \in [0, 1/2] \\ (0, 2t-1), & t \in [1/2, 1] \end{cases}$$

is a homeomorphism of the unit interval $[0, 1]$ and the "el" line (Fig. 29).

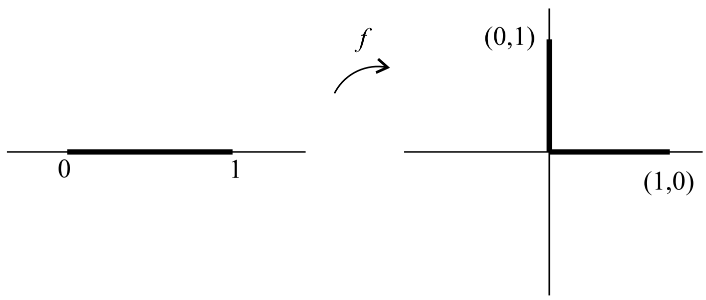

Fig. 29

8. The mapping $f\colon (0, 1) \to S^1 \setminus \{(1, 0)\}$, given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$ is a homeomorphism. Thus, we see that a circle without a point is homeomorphic to an open interval.

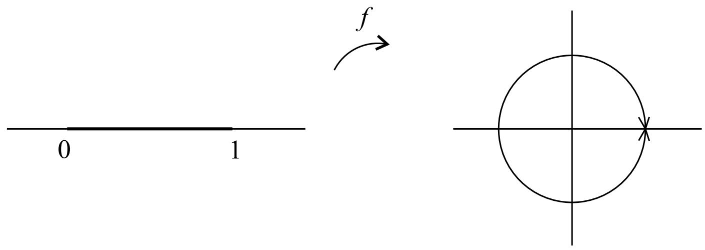

Fig. 30

9. Taking for domain the half closed interval $[0, 1)$ and for codomain the circle S^1, the formula $f(t) = (\cos 2\pi t, \sin 2\pi t)$ determines a mapping f, which is 1–1 and onto, continuous, but its inverse f^{-1} fails to be continuous. Indeed, let (A_n) be a sequence of points in S^1, converging to the point $(1, 0)$, as it is represented in Fig. 31. The inverse images $A'_n = f^{-1}(A_n)$, $n = 1, 2, \ldots$, scatter and the sequence (A'_n) does not converge to the point 0.

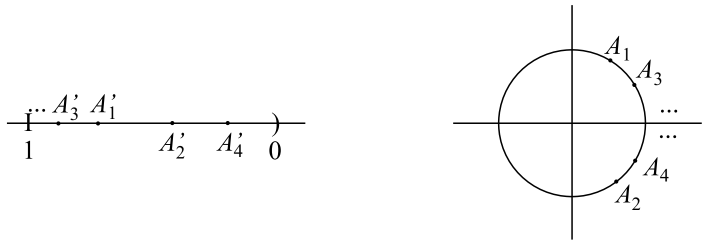

Fig. 31

We will see later that there is no mapping of the interval $[0, 1)$ onto the circle S^1, which would be a homeomorphism.

Exposing topological properties of geometric objects, we will consistently try to avoid mathematical formalism, preferring instead a perceptual basis of topological reasoning. A number of examples that we have just considered serve to convince the reader that there exists a formal, and therefore more rigorous, way of exposition.

When perceived, many objects of the outer world not only move, but they also bend, twist, turn, swell, shrink, etc. To extract then, the lasting and to differentiate it from the changing means the recognition of topological properties of such objects. Formally, a property is topological if preserved under homeomorphism.

Without cutting and tearing, an object stays as a whole and this is a fundamental topological property. Topological arc is taken as a prototype of this wholeness, which is formally expressed saying that arcs are connected objects. As a further step to generalization, a geometric object is said to be (arcwise) *connected* if for any pair of its different points there exists an arc joining them and being contained in that object. For instance, the objects:

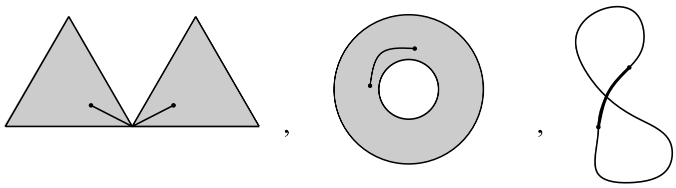

Fig. 32

are connected, while the following ones:

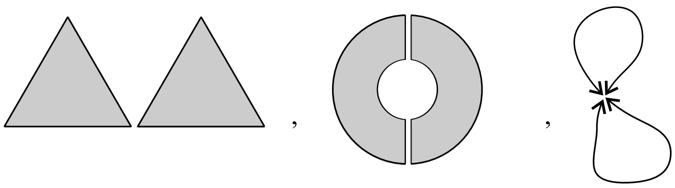

Fig. 33

are not (two triangles set one beside the other, ring with the points on an axis of symmetry removed, figure eight with the intersecting point removed).

When deforming, an arc joining two points deforms into arcs joining the same points. Under homeomorphisms, the image of an arc joining two points is again an arc joining their images. Hence, the property of an object to be connected is a topological property.

To prove that two objects are homeomorphic, we have to find a homeomorphism. To prove they are topologically different we have to find a topological property, which one of the objects have and the other one does not. "Good" objects are always connected and therefore, this property is not very discriminating. But there is a number from it derived properties, which are more efficient.

A point A of a geometric object X, is said to be *cutting* if, when removed, the object $X \setminus \{A\}$ is not connected. Then, $X \setminus \{A\}$ splits into a number of connected parts. The numbers of cutting and non-cutting points are topological properties. When a point A is cutting and the number of connected parts of $X \setminus \{A\}$ is m, then it is said that A *cuts* X *into* m *parts*, what is again a topological property. For example, the branching point of the "ef" line (Fig. 34) cuts it into 3 parts, the three end points are non-cutting and all other points cut it into 2 parts.

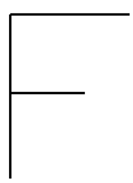

Fig. 34

9. All points x, $0 < x < 1$ of the interval $[0, 1)$ are cutting, while the circle S^1 has no cutting point. Hence, $[0, 1)$ and S^1 are not topologically equivalent and therefore, there is no mapping between these two objects which would be a homeomorphism.

All members of the following sequence of letter lines:

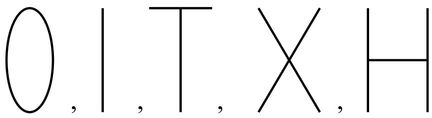

Fig. 35

are topologically different. Indeed, each of them has a topological property being not shared by any of the others. Following the order in which the lines stand, such properties are:

- the "O" line has no cutting point (all its points are non-cutting),
- the lines "I", "T", "X" have 2, 3 and 4 non-cutting points, respectively,
- the "X" line has a point cutting it into 4 parts and the "H" line does not. Having 4 non-cutting points, the "H" line is also different from the first three lines.

10. The two lines on Fig. 36 have no cutting points. Analogously to the case of cutting point, a finite set of points which cuts an object into a number of parts can also be employed for topological discrimination. Thus, the first line has a 2 point set which cuts it into 3 parts, while the second one does not have. This example could inspire the interested reader to do several similar, topological comparisons of lines.

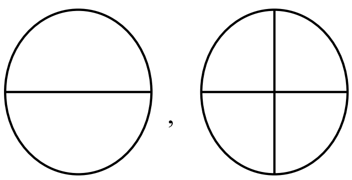

Fig. 36

Now we will list a number of the properties of geometric objects, which are topological in nature and which appear even at the early stage of teaching geometry.

Looking at a line locally, that is considering it only in small neighbourhoods of its points, the line extends in only one direction. When looking locally, a surface extends in two directions and a solid body in three. The number of directions in which a geometric object extends locally is a visual ground upon which the topological concept of dimension is laid.

In topology, a line is one-, a surface two- and a solid body three-dimensional object. Deforming a line it stays to be a line, a surface to be a surface and a solid body to be a solid body. Thus, the properties of a geometric object to be a line, a surface or a solid body are topological.

As we have already seen, a topological arc and a topological circle are topologically distinct objects. In early school geometry they are called an open and a closed curve, respectively and they represent purely topological concepts.

To be an end point of a line (that is a non-cutting point) is again a topological property.

A closed curve splits the plane into two regions, one being inside and the other one outside that curve. This important topological property of closed curves, which the eye comprehends in one sweep, is rather difficult to be proved mathematically. The relationship of a point and of a closed curve is of three kinds—a point can be in, on or out of a closed curve. Thus, we see that the meaning of the prepositions "*in*", "*on*" and "*out*" is also topological.

When we say that a point is between the other two, we imagine the points to be collinear and such a concept is projective. But when we have a configuration—an open line with three points on it, then one of the points is between the other two, giving to this relationship a larger sense than before. Since such a relationship does not change under deformations, it is topological in nature.

Notice that the number of intersecting points of two lines and boundaries of geometric objects are some further topological concepts. And we are sure that an enquiring reader will find still other topological properties and concepts involved in the subject matter of school geometry course.

Let us end this exposition with a remark showing the hierarchy of the kinds of all considered properties. A projection is a homeomorphism of one plane onto another. Thus, projections preserve all topological properties or, expressing it in other words, each topological property of an object is also its projective property. And, as already seen, a projective property is Euclidean and an Euclidean one is metric.

6. Euler-Poincare characteristic

If you have been intrigued by the contents of the previous section, then you will certainly enjoy to read this one, as well. Here we expose a fascinating topological property, which is expressed in numbers and which aids us in distinguishing topologically one object from another when such objects are more complex than the lines are. The idea has its roots in a property of polyhedral surfaces that you probably know from your school geometry courses. Namely, when we take the surfaces of polyhedra, as a pyramid or a prism is:

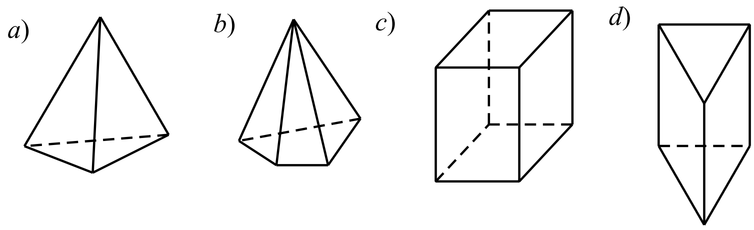

Fig. 37

and when we form the alternating sum:

$$v - e + f,$$

where v denotes the number of vertices, e the number of edges and f the number of faces, then in each of these cases the number 2 results. Indeed,

a) $4 - 6 + 4 = 2$, b) $5 - 8 + 5 = 2$, c) $8 - 12 + 6 = 2$, d) $6 - 9 + 5 = 2$.

In the general case of a polyhedral surface which is a topological sphere, the equality $v - e + f = 2$ holds. This fact was probably known in antiquity, but in the contemporary geometry it is referred to as the Euler theorem (after the great classic mathematician, Leonhard Euler (1707–1783)).

A powerful generalization of the Euler theorem is obtained when such alternating sums are formed for higher dimensional polyhedra and especially, proving that so resulting numbers are topological properties of geometric objects themselves, independently existing of the ways how they are represented as polyhedra. Such numbers are called the Euler-Poincare characteristics and the mentioned generalization was carried out by, the great mathematician Henri Poincare (1854–1912), who is also considered to be the founder of topology as an independent branch of mathematics.

For example, the topological circle S^1 can be differently represented as a polygonal line:

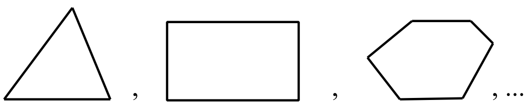

Fig. 38

and all sums of the form $v - e$:

$$3 - 3 = 0, \quad 4 - 4 = 0, \quad 6 - 6 = 0, \quad \dots$$

give the number 0, as a topological property of S^1. When the topological disk is represented as a triangle, a quadrilateral, a hexagon, ..., the corresponding sum $v - e + f$ will be:

$$3 - 3 + 1 = 1, \quad 4 - 4 + 1 = 1, \quad 6 - 6 + 1 = 1, \quad \dots$$

and the number 1 is a topological property of the disk. On the basis of this property, the circle and the disk are topologically different objects.

Proceeding further, we will expose a method of calculation of Euler-Poincare characteristic simpler than that which has already been used for polyhedral surfaces. We will not try to prove that this characteristics are topological properties, since such a proof depends on very sophisticated techniques.

Now we start describing the method, step by step. Denoting a geometric object with X, its Euler-Poincare characteristic will be denoted with $\chi(X)$.

When X is one point, $\chi(X) = 1$ (just one vertex). When X consists of n points, then $\chi(X) = n$ and more generally, when the objects X_i, $i = 1, \dots, n$ do not intersect, then

$$\chi\Big(\bigcup X_i\Big) = \chi(X_1) + \cdots + \chi(X_n).$$

To calculate $\chi(X)$, when X is a line, such an object has to be seen in fibres, which are finite sets of points. In Fig. 39, we see a topological arc and topological circle in fibres.

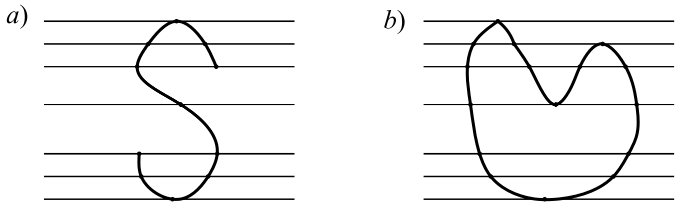

Fig. 39

In the case a), the fibres are: f_0—one point, f_1—running two points, f_2—two points, f_3—running one point, f_4—two points, f_5—running two points and f_6—one point. Then, $\chi(X)$ is calculated this way

$$\begin{aligned}\chi(X) &= \chi(f_0) - \chi(f_1) + \chi(f_2) - \chi(f_3) + \chi(f_4) - \chi(f_5) + \chi(f_6)\\ &= 1 - 2 + 2 - 1 + 2 - 2 + 1 = 1.\end{aligned}$$

In the case b), the fibres are (looking upwards): f_0—one point, f_1—running two points, f_2—three points, f_3—running four points, f_4—three points, f_5—running two points and f_6—one point. Then, as before

$$\chi(X) = 1 - 2 + 3 - 4 + 3 - 2 + 1 = 0.$$

First we have the initial fibre f_0, then the running fibres f_1, which are all topologically equivalent each to other, then the fibre f_2, which is topologically different from f_1 (the case b)) or the running fibres f_3 following it are topologically different from f_2 (the case a)) and so on and so forth.

Using the Gestalt language, the two objects in Fig. 39 are "poor" forms. Their "good" forms are seen in the following figure, when the calculation is also easier.

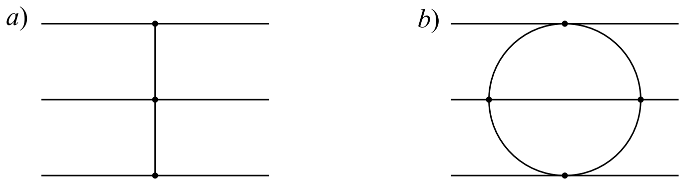

Fig. 40

Case a): $\chi(X) = 1-1+1 = 1$, case b): $\chi(X) = 1-2+1 = 0$. The two "poor" forms were intentionally used for explanation of the way how an object is decomposed into fibres, as well as, to demonstrate the independence of a topological property of the shape of objects.

It is known that children easily identify a badly distorted disk with, say, two holes with the "good" one (Fig. 41), grasping the number of holes as a common property. Now we will show that the number of holes is, indeed, a discriminating topological property, but instead of "poor" we will use "good" forms.

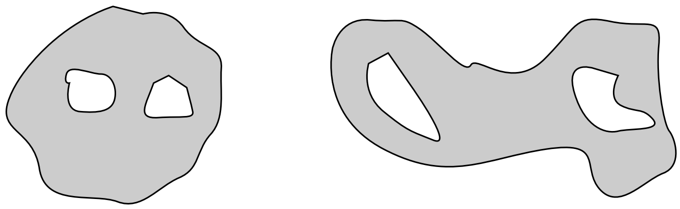

Fig. 41

1. Let us consider a disk, a disk with one and a disc with two holes.

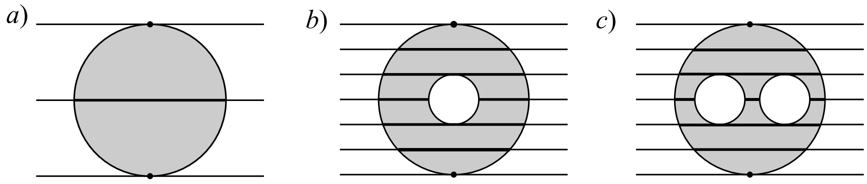

Fig. 42

$$\text{a) } \chi(X) = 1 - 1 + 1 = 1, \qquad \text{b) } \chi(X) = 1 - 1 + 1 - 2 + 1 - 1 + 1 = 0,$$
$$\text{c) } \chi(X) = 1 - 1 + 1 - 3 + 1 - 1 + 1 = -1.$$

We suggest to the reader to prove that, when X is a disk with n holes, then $\chi(X) = -n + 1$.

2. The results obtained in this example will be used in the one, which follows it. We consider here two, three, ... touching circles as it is illustrated in Fig. 43.

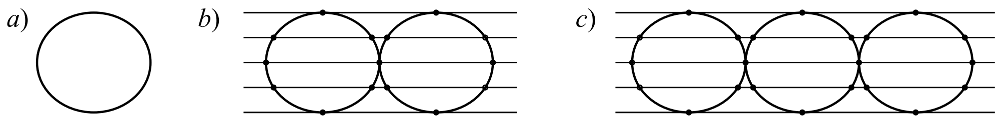

Fig. 43

$$\text{a) } \chi(X) = 0, \ \text{b) } \chi(X) = 2 - 4 + 3 - 4 + 2 = -1, \ \text{c) } \chi(X) = 3 - 6 + 4 - 6 + 3 = -2$$

Again, we suggest to the reader to prove that, when X is the line consisting of n circles touching in the way illustrated in Fig. 43, then $\chi(X) = 1 - n$.

3. The surfaces that we consider now are a sphere, a sphere with one hole (also called a torus), a sphere with two holes, ... Representations in fibres are obtained by intersection with a family of parallel planes (Fig. 44).

$$\text{a) } \chi(X) = 1 - 0 + 1 = 2, \quad \text{b) } \chi(X) = 1 - 0 + (-1) - 0 + (-1) - 0 + 1 = 0$$
$$\text{c) } \chi(X) = 1 - 0 + (-2) - 0 - (-2) - 0 + 1 = -2, \ldots$$

We suggest to the reader to prove that, when X is a sphere with n holes, then $\chi(X) = 2 - 2n$.

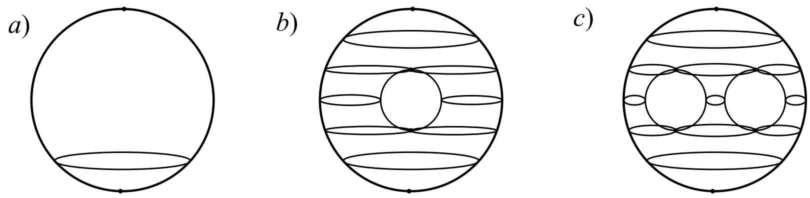

Fig. 44

In the case of these surfaces we see again that the number of holes is a significant topological property.

We end this exposition with a number of remarks. Some surfaces have lines as their boundaries: a half-sphere a circle, a cylinder two circles, etc. Sphere and spheres with a number of holes have no boundary. An intersecting fact for your information will be to know that there is no other orientable surface without boundary, which would be topologically distinct from a sphere or spheres with holes. And a surface is orientable if it is two-sided—when without boundary it has an inner and an outer side and when with boundary, a path from one side to the other must go across the boundary. But there exist surfaces, which are one-sided. A very popular example is the Möbius band (Fig. 45, a)), whose paper model is easily made when a rectangular paper band is twisted first and then two opposite edges glued together. Another example is seen in Fig. 45, b)—a rectangular paper with two holes is bent first and then two ends of a pipe are glued to the rims of the holes.

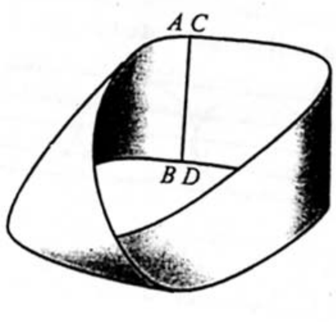

Fig. 45a

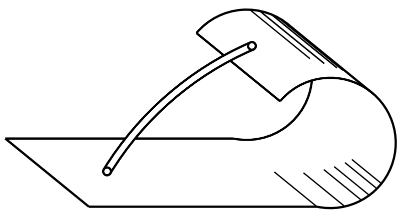

Fig. 45b

If you are interested to know more about projective geometry or topology, we direct you to the excellent book of Courant, Robins [3].

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