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SELECTED CHAPTERS FROM ALGEBRA

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Abstract. This paper is the seventh (and last) part of the publication "Selected chapters from algebra", the first six having been published in previous issues of the Teaching of Mathematics, Vol. I (1998), 1–22, Vol. II, 1 (1999), 1–30, Vol. II, 2 (1999), 65–80, Vol. III, 1 (2000), 15–40, Vol. III, 2 (2000), 63–82, Vol. IV, 1 (2001), 1–34 and Vol. IV, 2 (2001), 71–104.

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CHAPTER VII. POWER SERIES

1. Polynomials as generating functions

We have often met the fact that some properties of a finite sequence of numbers $(a_0, a_1, \dots, a_n)$ can be described using the polynomial $f(x) = a_0 + a_1x + \dots + a_nx^n$. The polynomial $f(x)$ is called the *generating function* of the sequence $(a_0, a_1, \dots, a_n)$. A very nice example of this kind is the question considered in Chapter III, when a finite set M was given and a_k was the number of its subsets which contain k elements. If we introduce the generating function $f_M(x) = a_0+a_1x+\dots+a_nx^n$, then the value a_k for the subset M_1+M_2 is expressed by a very simple formula $f_{M_1+M_2}(x) = f_{M_1}(x)f_{M_2}(x)$ (formula (8) in Ch. III).

In the same way, the binomial coefficients C_n^k $(k = 0, 1, \dots, n)$ can be explored conveniently using the generating function $f(x) = (1 + x)^n$. Many identities for binomial coefficients can be easily deduced from this fact (see formula (26) of Ch. II and Problem 5 in Sec. 3 of Ch. II).

We shall now give some more examples of similar kind.

The first example is concerned with properties of natural numbers. We consider representations of a natural number n as a sum of natural numbers: $n = a_0 + a_1 + \dots + a_k$. A representation of this kind will be called a *partition* of the number n. Two partitions will be considered equal if their summands $(a_0, a_1, \dots, a_n)$ are equal,

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possibly taken in different order. E.g., the representations $6 = 1+1+4 = 1+4+1 = 4+1+1$ will be considered as the same partition of number 6.

Denote by $P_{k,l}(n)$ the number of partitions of number n into not more than k summands, each of which does not exceed l. In order to explore the numbers $P_{k,l}(n)$ we shall construct a generating function. Define $P_{0,0}(0) = 1$. Note that if there exists at least one partition of n satisfying the given conditions then $n \leqslant kl$. Hence, we can form the sum of all expressions $P_{k,l}(n)x^n$ with $n = 0, 1, 2, \dots$; only numbers with $n \leqslant kl$ will enter this sum, and so it will be a polynomial. We shall denote this polynomial by $g_{k,l}(x)$:

$$g_{k,l}(x) = P_{k,l}(0) + P_{k,l}(1)x + \cdots + P_{k,l}(kl)x^{kl}. \tag{1}$$

Obviously, $g_{0,l}(x) = 1$ and $g_{k,0}(x) = 1$. We now deduce two relations connecting the polynomials $g_{k,l}$ with the polynomials of the same kind but with smaller indices. Consider the difference $P_{k,l}(n) - P_{k,l-1}(n)$. The first summand is equal to the number of partitions of n to k summands not exceeding l, $n = a_1 + \cdots + a_j$, $j \leqslant k$, $a_i \leqslant l$, and the second—not exceeding $l-1$. Obviously, the difference is equal to the number of partitions $n = a_1+\cdots+a_j$, where j again does not exceed k, $a_i \leqslant l$ and at least one of $a_1, \dots, a_j$ is equal to l, e.g., $a_1 = l$. Eliminating this summand, we obtain a partition of $n-l$: $n-l = a_2+\cdots+a_j$, where the number of summands now does not exceed $k-1$, and the summands do not exceed l as before. In this way, we obtain a one-to-one correspondence between $P_{k,l}(n) - P_{k,l-1}(n)$ partitions of number n and $P_{k-1,l}(n-l)$ partitions of number $n-l$. Therefore,

$$P_{k,l}(n) - P_{k,l-1}(n) = P_{k-1,l}(n-l). \tag{2}$$

The number $P_{k-1,l}(n-l)$ is by definition equal to the coefficient of x^{n-l} in the polynomial $g_{k-1,l}(x)$, and hence to the coefficient of x^n in the polynomial $g_{k-1,l}(x)x^l$. Therefore, relation (2) gives us the equality of coefficients of x^n in the polynomials $g_{k,l} - g_{k,l-1}$ and $g_{k-1,l}x^l$. Since it is valid for each n, we obtain that

$$g_{k,l}(x) = g_{k,l-1}(x) + g_{k-1,l}(x)x^l. \tag{3}$$

The second relation is derived completely analogously. Consider the difference $P_{k,l}(n) - P_{k-1,l}(n)$. The first summand is equal to the number of partitions of n into no more than k summands not exceeding l, and the second to the number of partitions of n into no more than $k-1$ summands of the same kind. Thus, the difference expresses the number of partitions $n = a_1 + \cdots + a_k$ into exactly k natural summands, not exceeding l. If we subtract 1 from each summand, and if a particular summand is equal to 1, then eliminate this difference. As a result we obtain the partition $n-k = b_1 + \cdots + b_j$, where $j \leqslant k$, $b_i \leqslant l-1$ and it is obvious that the difference $P_{k,l}(n) - P_{k-1,l}(n)$ is equal to the number of these partitions. In other words, we have proved that

$$P_{k,l}(n) - P_{k-1,l}(n) = P_{k,l-1}(n-k).$$

As before, it follows that

$$g_{k,l}(x) = g_{k-1,l}(x) + g_{k,l-1}(x)x^k. \tag{4}$$

Relations (3) and (4) enable us to find the explicit formula for the polynomial $g_{k,l}(x)$. It follows from them that

$$g_{k,l-1}(x) + g_{k-1,l}(x)x^l = g_{k-1,l}(x) + g_{k,l-1}(x)x^k,$$

wherefrom $g_{k,l-1}(x)(1-x^k) = g_{k-1,l}(x)(1-x^l)$, and so

$$g_{k,l-1}(x) = g_{k-1,l}(x)\frac{1-x^l}{1-x^k}.$$

Replacing in this relation l by $l+1$, we obtain

$$g_{k,l}(x) = g_{k-1,l+1}(x)\frac{1-x^{l+1}}{1-x^k}. \tag{5}$$

Relation (5) can now be applied to the polynomial $g_{k-1,l+1}(x)$, and as a result we obtain

$$g_{k,l}(x) = g_{k-2,l+2}(x)\frac{1-x^{l+1}}{1-x^k}\frac{1-x^{l+2}}{1-x^{k-1}}.$$

The process can be repeated k times, and since $g_{0,k+l}(x) = 1$, we finally obtain the formula

$$g_{k,l}(x) = \frac{(1-x^{l+1})(1-x^{l+2})\cdots(1-x^{l+k})}{(1-x^k)(1-x^{k-1})\cdots(1-x)}. \tag{6}$$

Formula (6) acquires a more symmetrical form if on the right-hand side both the numerator and the denominator are multiplied by $(1-x)(1-x^2)\dots(1-x^l)$. If we denote the polynomial $(1-x)(1-x^2)\cdots(1-x^m)$ by $h_m(x)$, formula (6) acquires the form

$$g_{k,l}(x) = \frac{h_{k+l}(x)}{h_k(x)h_l(x)}. \tag{7}$$

The expression on the right-hand side has the structure analogous to the binomial coefficient C_{k+l}^k, while polynomial $h_k(x)$ is the analogue of the number $k!$. Polynomials $g_{k,l}(x)$ defined by equality (7) are called *Gauss polynomials*. As in the case of binomial coefficients, it is not immediately clear that the fraction $\dfrac{h_{k+l}(x)}{h_k(x)h_l(x)}$ is a polynomial. It follows, of course, from the connection of the polynomial $g_{k,l}(x)$ with partitions, i.e., from its definition by formula (1) (see, however, Problem 3).

We shall deduce now some properties of polynomials $g_{k,l}(x)$ which are analogous to the known properties of binomial coefficients. It follows obviously from formula (7) that

$$g_{k,l}(x) = g_{l,k}(x), \tag{8}$$

analogously to the property of binomial coefficients $C_{k+l}^k = C_{k+l}^l$ (since polynomials $g_{k,l}(x)$ are analogous to coefficients C_{k+l}^k). Relations (3) and (4) can be transformed into each other by application of relation (8). Putting $g_{l,k} = g_{k,l}$, relation (3) gives $g_{l,k} = g_{l,k+1} + g_{l-1,k}x^k$ and then, again using relation (8), $g_{l,k-1} = g_{k-1,l}$, $g_{l-1,k} = g_{k,l-1}$ and one obtains relation (4) from relation (3). Both of these relations are analogous to the equality $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$—formula (26) in Ch. II (n has to replaced by $k+l$; then C_{n-1}^k is obtained from C_n^k replacing l with $l-1$ and C_{n-1}^{k-1} replacing k with $k+1$). Finally, a direct connection (and not analogy) with binomial coefficients follows from the relation

$$g_{k,l}(1) = C_{k+l}^k. \tag{9}$$

Direct substitution $x = 1$ in (6) is not possible—numerator and denominator would become 0. We need to divide both the numerator and denominator by $(1-x)^k$, or, more precisely, divide each factor entering numerator and denominator by $1-x$. The polynomial $1-x^m$ is divisible by $1-x$ for each m and $\dfrac{1-x^m}{1-x} = 1 + x + \cdots + x^{m-1}$ (formula (12) in Ch. I). Hence, $\dfrac{1-x^m}{1-x}(1) = m$. Dividing each factor from numerator and denominator of formula (6) and substituting $x = 1$, we obtain

$$g_{k,l}(1) = \frac{(l+k)\cdots(l+2)(l+1)}{1\cdot 2\cdot\ldots\cdot k}$$

(we have written down factors in both numerator and denominator in reverse order). This shows that $g_{k,l}(1) = C_{k+l}^k$.

Finally, let us demonstrate an important property of Gauss polynomials $g_{k,l}(x)$ which does not have an analogue for binomial coefficients. Bearing in mind the fact that $g_{k,l}(x)$ are polynomials and not numbers, we shall prove that the polynomial $g_{k,l}(x)$ is *reciprocal* for each k and l. Recall (Sec. 3, Ch. III) that the polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ of degree n is called reciprocal if the coefficients equidistant from the ends are equal, i.e., $a_k = a_{n-k}$ for $k = 0, 1, \ldots, n$. A polynomial $f(x)$ of degree n is reciprocal if and only if $x^nf(1/x) = f(x)$ (this was also proved in Sec. 3, Ch. III). The polynomial $g_{k,l}(x)$ has degree kl: it follows from its representation (1) and from $P_{k,l}(kl) \geqslant 1$: there exists at least one partition of kl into k summands being equal to l: $kl = l + \cdots + l$ (k times) (the same can be deduced easily from representation (6) if one calculates the powers of numerator and denominator and subtract the latter from the first). Thus, we have only to check the relation $x^{kl}g_{k,l}(\frac{1}{x}) = g_{k,l}(x)$. This follows immediately from (6). Note that for arbitrary m the equality $\left(1 - \dfrac{1}{x^m}\right) = (-1)x^{-m}(1-x^m)$ is valid. Hence, the relation of this kind is valid for each factor both in the numerator and in the denominator on the right-hand side of formula (6). Since the number of factors in numerator and denominator is the same (it is equal to k), all of the factors (-1) will cancel out. The factor x^{-m} can be taken out of each factor $1-x^m$ of the degree m. We obtain that $g_{k,l}(\frac{1}{x}) = x^{-N}g_{k,l}(x)$, where N is the difference of degrees of the numerator and denominator. But this difference is equal to the degree of the polynomial $g_{k,l}(x)$, i.e., to kl. Hence, $N = kl$, $g_{k,l}(\frac{1}{x}) = x^{-kl}g_{k,l}(x)$ and so $x^{kl}g_{k,l}(\frac{1}{x}) = g_{k,l}(x)$ and this means that polynomial $g_{k,l}$ is reciprocal.

The properties of Gauss polynomials we have deduced imply the corresponding properties of partitions (more precisely, of numbers $P_{k,l}(n)$, if we pass to their coefficients using the definition (1)). For example, relation (8) gives the equality

$$P_{k,l}(n) = P_{l,k}(n), \tag{10}$$

i.e., *the number of partitions of n into at most k summands not exceeding l is equal to the number of its partitions into at most l summands not exceeding k.* Reciprocality of the polynomial $p_{k,l}(x)$ implies that

$$P_{k,l}(n) = P_{k,l}(kl - n). \tag{11}$$

Relation (9) means that for given numbers k and l the sum of all numbers $P_{k,l}(n)$ for $n = 0, 1, \dots, kl$ is equal to C^k_{k+l}, i.e.,

$$P_{k,l}(0) + P_{k,l}(1) + \cdots + P_{k,l}(kl) = C^k_{k+l}. \tag{12}$$

Of course, these easy properties of partitions can be proved without using the generating functions $g_{k,l}(x)$ (see Problems 4, 5, 6). But the easiest way to discover them is by the use of generating functions.

Finally, note that in Section 3 of Chapter III we considered one more property of polynomials—unimodality. For a reciprocal polynomial $a_0 + a_1x + \cdots + a_Nx^N$ unimodality means that $a_i \leqslant a_{i+1}$ for $i + 1 \leqslant N/2$. Then from reciprocality it follows that $a_j \geqslant a_{j+1}$ for $j \geqslant N/2$. It turns out that the Gauss polynomials $g_{k,l}(x)$ have the property of unimodality. By definition it means that

$$P_{k,l}(n) \leqslant P_{k,l}(n+1) \quad \text{for } n + 1 \leqslant \frac{kl}{2}.$$

The only known proof of this fact is based on a connection of numbers $P_{k,l}(n)$ with a completely different section of Algebra. Namely, the number $P_{k,l}(n+1) - P_{k,l}(n)$ for $n+1 \leqslant kl/2$ coincides with the number of elements of a certain finite set, and is therefore nonnegative. As specialists assure us, there is no known "natural" proof of this fact, based on the properties of partitions or polynomials $g_{k,l}(x)$. Maybe some of the readers of this book will succeed in finding such a proof.

As the *second example* we shall consider some well known properties of natural numbers which can be deduced in an elegant way with the use of generating functions. We are talking about the ability to write down all natural numbers in a number system with the given base.

Let us start with the binary system. For an arbitrary natural number n, one can find the largest power of 2 dividing this number, and so it can be represented in the form $n = 2^k m$, where m is odd. As m has the form $2r + 1$, n can be represented as $n = 2^k + 2^{k+1}r$. Now the same reasoning can be applied to number r, and continuing the process we finally represent n in the form of a sum of *distinct* powers of 2: $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_m}$, where $k_1 > k_2 > \cdots > k_m$. In other words, we represent n in the form

$$n = a_0 + a_1 2 + a_2 2^2 + \cdots + a_N 2^N, \tag{13}$$

where the coefficients $a_0, a_1, \dots, a_N$ can take values 0 and 1. Eliminating the terms for which $a_i = 0$, we return to the representation of n as the sum of distinct powers of 2. The representation (13) is called the *binary representation* of the number n, or the representation of n in the *binary system.* Let us prove that representation (13) is, for the given number n, unique. Let $n = b_0 + b_1 2 + \cdots + b_M 2^M$ be another representation. Then $a_0 = b_0$ because if n is odd, then $a_0 = b_0 = 1$, and if n is even, then $a_0 = b_0 = 0$. Hence, in the second representation we can put $b_0 = a_0$ and obtain that $\frac{n-a_0}{2} = a_1 + a_2 2 + \cdots + a_N 2^{N-1}$, $\frac{n-a_0}{2} = b_1 + b_2 2 + \cdots + b_M 2^{M-1}$. Since $\frac{n-a_0}{2} \leqslant \frac{n}{2} < n$, we obtained two different representations for the number $\frac{n-a_0}{2}$ which is smaller than n. Using induction, we could assume that our assertion was valid for $\frac{n-a_0}{2}$, and so $a_1 = b_1$, $a_2 = b_2$, etc. (Besides, the reader had probably proved this before, solving Problem 5 of Sec. 1, Ch. I).

For the given value of N, we obtain the largest number n in representation (13) when the numbers a_i take the largest possible value, i.e., when all $a_i = 1$ and $n = 1 + 2 + \cdots + 2^N = \dfrac{2^{N+1} - 1}{2 - 1} = 2^{N+1} - 1$. Hence, for the given value of N, all numbers smaller than 2^{N+1}, and just them, can be written in the form (13), and the corresponding representation is unique.

On the other hand, consider the product

$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \cdots (1 + x^{2^N}). \tag{14}$$

Expanding the brackets, we have to take one term from each bracket, i.e., from $(1 + x^{2^i})$ we take either 1 or x^{2^i}. As a result, we obtain the term in x^n, where n is the sum of distinct powers of 2, i.e., sum of numbers 2^i for certain $i \leqslant N$. As we have seen, each number $n \leqslant 2^{N+1} - 1$ can be obtained in this way and, moreover, each number exactly once. Hence, expanding the brackets in the product (14), we obtain all terms x^n with $n \leqslant 2^{N+1} - 1$ with coefficient 1. In other words, the assertion that each number $n \leqslant 2^{N+1} - 1$ has a unique binary representation gives us the identity

$$(1 + x)(1 + x^2)(1 + x^4) \cdots (1 + x^{2^N}) = 1 + x + x^2 + x^3 + \cdots + x^{2^{N+1}-1}. \tag{15}$$

One can easily verify that our reasoning can be followed in reverse order, i.e., identity (15) implies the existence of unique binary representations for all numbers $n \leqslant 2^{N+1} - 1$.

How can we show directly that relation (15) is valid and so prove again the existence and uniqueness of binary representation? The right-hand side of equality (15) can be transformed using the familiar formula

$$1 + x + x^2 + x^3 + \cdots + x^{2^{N+1}-1} = \frac{1 - x^{2^{N+1}}}{1 - x}.$$

Thus, in order to prove identity (15), it is sufficient to check the identity

$$(1 - x)(1 + x)(1 + x^2)(1 + x^4) \cdots (1 + x^{2^N}) = 1 - x^{2^{N+1}}.$$

But this is obvious! Multiplying the first two factors, we obtain $1 - x^2$. Multiplying $(1 - x^2)$ by $(1 + x^2)$, we obtain $1 - x^4$, etc., until, finally, multiplying $(1 - x^{2^N})$ by $(1 + x^{2^N})$ we obtain $1 - x^{2^{N+1}}$.

Consider now a completely analogous case of the decimal system. Divide with remainder an arbitrary natural number n by 10: $n = 10n_1 + a_0$, where $0 \leqslant a_0 \leqslant 9$. Then, divide with remainder n_1 by 10: $n_1 = 10n_2 + a_1$, where $0 \leqslant a_1 \leqslant 9$. Substituting, we obtain that $n = 10^2 n_2 + 10n_1 + a_0$. Continuing this process, we obtain for some k that $n = 10^k a_k + 10^{k-1}a_{k-1} + \cdots + 10a_1 + a_0$, where $0 \leqslant a_i \leqslant 9$ for all a_i. This is our usual decimal representation of the number n. It is unique. Really, writing down the same formula in the form $n = 10(10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1) + a_0$, i.e., $n = 10m + a_0$, where $m = 10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1$, we see that a_0 is the remainder of division of n by 10. But, division with remainder is unique (Theorem 4, Ch. I). Therefore, if distinct decimal representations existed, all of them would have at least the same a_0—it is equal to the remainder of division of number n by 10. If $n = 10^l b_l + 10^{l-1}b_{l-1} + \cdots + 10b_1 + b_0$ were another decimal representation, where $0 \leqslant b_i \leqslant 9$ for all b_i, then we could assert that $a_0 = b_0$. Thus,

$$10^k a_k + 10^{k-1}a_{k-1} + \cdots + 10a_1 = 10^l b_l + 10^{l-1}b_{l-1} + \cdots + 10b_1.$$

Dividing by 10, we obtain that

$$10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1 = 10^{l-1}b_l + 10^{l-2}b_{l-1} + \cdots + b_1,$$

i.e., we have two decimal representations of the number $m = \dfrac{n - a_0}{10}$. Since $m \leqslant \dfrac{n}{10} < n$, we can, using induction, assume that m has a unique decimal representation, and this means that $a_1 = b_1$, $a_2 = b_2$, etc.

It is clear that the number n with the decimal representation $n = 10^k a_k + 10^{k-1}a_{k-1} + \cdots + 10a_1 + a_0$ does not exceed $9(10^k + 10^{k-1} + \cdots + 10 + 1)$ (since all $a_i \leqslant 9$), and the last number is equal to $9 \cdot \dfrac{10^{k+1} - 1}{10 - 1} = 10^{k+1} - 1$. Thus, all the numbers not exceeding $10^{k+1} - 1$ (i.e., being less than 10^{k+1}) have the decimal representation of the form $10^k a_k + 10^{k-1}a_{k-1} + \cdots + 10a_1 + a_0$ with the given k, and they are the only numbers that have this property. Let us write down this fact as an identity between polynomials. Consider the product $(1 + x + x^2 + \cdots + x^9)(1 + x^{10} + x^{20} + \cdots + x^{90}) \cdots (1 + x^{10^k} + x^{2\cdot 10^k} + \cdots + x^{9\cdot 10^k})$. Removing the brackets, we have to take from the first bracket a factor x^{a_0}, where a_0 takes one of the values 0, 1, … , 9; from the second bracket—a factor x^{10a_1}, where a_1 takes one of the same values, and similarly for the rest of the brackets. As a result, we obtain the term $x^{a_0 + 10a_1 + \cdots + 10^k a_k}$, and this is, as we have seen, an arbitrary term x^n, where n is any number not exceeding $10^{k+1} - 1$—this is just the assertion of the existence of decimal representation. Each term of this kind will be obtained just once, i.e., with the coefficient 1. Thus, the existence and uniqueness of decimal expansion of the form $n = 10^k a_k + 10^{k-1}a_{k-1} + \cdots + 10a_1 + a_0$ for all numbers $n \leqslant 10^{k+1} - 1$ imply the identity

$$(16) \quad (1 + x + x^2 + \cdots + x^9)(1 + x^{10} + x^{20} + \cdots + x^{90}) \cdots \\ \cdots (1 + x^{10^k} + x^{2\cdot 10^k} + \cdots + x^{9\cdot 10^k}) = 1 + x + x^2 + \cdots + x^{10^{k+1}-1}.$$

As in the case of binary representation, the whole argument can be made in the reverse order, and so, conversely, identity (16) implies the existence and uniqueness of decimal representation. Let us try to prove identity (16) directly, and hence to prove again the existence and uniqueness of decimal representation. This is very easy. Apply again the formula

$$1 + x + x^2 + \cdots + x^{10^{k+1}-1} = \frac{x^{10^{k+1}-1}}{x-1}$$

to the right-hand side of relation (16). Transform each factor in the brackets on the left-hand side in an analogous manner:

$$\begin{aligned} 1 + x + x^2 + \cdots + x^9 &= \frac{x^{10}-1}{x-1}, \\ 1 + x^{10} + x^{20} + \cdots + x^{90} &= \frac{x^{100}-1}{x^{10}-1}, \\ &\cdots\cdots \\ 1 + x^{10^k} + x^{2\cdot 10^k} + \cdots + x^{9\cdot 10^k} &= \frac{x^{10^{k+1}}-1}{x^{10^k}-1}. \end{aligned}$$

Relation (16) then takes the form

$$\frac{x^{10}-1}{x-1}\frac{x^{100}-1}{x^{10}-1}\cdots\frac{x^{10^{k+1}}-1}{x^{10^k}-1} = \frac{x^{10^{k+1}}-1}{x-1}.$$

This is completely obvious: on the left-hand side the numerator of each factor cancels out with the denominator of the next factor and there only remains $\dfrac{1}{x-1}$ (from the first factor) and $x^{10^{k+1}} - 1$ (from the last one).

Number systems with other bases can be considered in the same way.

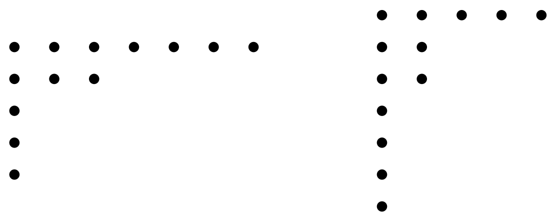
Problems

1. Find the explicit form of polynomials $g_{k,1}(x)$, starting from their definition and from formula (6).

2. Find the explicit form of polynomials $g_{k,2}(x)$. (It is a bit more complicated than Problem 1.)

3. Let the rational expressions $g_{k,l}(x)$ be *defined* by formula (7). Prove that they satisfy relations (3) and (4) and hence prove that they are polynomials (not using formula (1) and its connection with the theory of partitions).

4. Prove that $P_{k,l}(n) = P_{l,k}(n)$ without using properties of Gauss polynomials. [*Hint*: The partition $n = a_1 + \cdots + a_j$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_j$, can be represented by a table of points, having a_1 points in the first row, a_2 points in the second, etc. For example, the partition $13 = 7 + 3 + 1 + 1 + 1$ can be represented by the first table in the next figure



Correspond to each table the "turned" table, rows of which are columns of the original one. E.g., to the previous table, there will correspond the second one in the previous figure.]

5. Prove that $P_{l,k}(n) = P_{l,k}(kl-n)$ without using properties of Gauss polynomials. [*Hint*: Correspond to the partition $n = a_1 + \cdots + a_j$, $j \leqslant k$, $a_i \leqslant l$, of the number n, the partition $kl - n = (l - a_1) + (l - a_2) + \cdots + (l - a_j) + l + \cdots + l$ of the number $kl - n$, where the summand $l - a_i$ is eliminated if it is equal to 0, and the number of summands equal to l is equal to $k - j$.]

6. Prove that $P_{l,k}(0)+P_{l,k}(1)+\cdots+P_{l,k}(kl) = C^k_{k+l}$ without using properties of Gauss polynomials. [*Hint*: Correspond to the partition $a_1 + \cdots + a_j$, $j \leqslant k$, $a_i \leqslant l$, of a number not exceeding kl, the subset $\{a_1 + 1, a_2 + 2, \dots, a_j + j\}$ of the set $\{1, 2, \dots, k + l\}$.]

7. Prove that each weight of a whole number of kilograms and smaller than 2^n, can be determined using n weights of 1, 2, 2^2, ... , 2^{n-1} kilograms (the measured object is placed on one side of the scales and weights on the other side).

8. It is allowed to put weights on both sides of the scales in this. Prove that an arbitrary weight of a whole number of kilograms and smaller than $\frac{3^n-1}{2}$ can be determined using n weights of 1, 3, ... , 3^{n-1} kilograms. [*Hint*: Prove the existence and uniqueness of ternary representation of the form $m = a_0+a_1 3+\cdots+a_{n-1}3^{n-1}$, where a_0, a_1, ... , a_{n-1} take values 1, 0 or -1, for all integers m between $-\frac{3^n-1}{2}$ and $\frac{3^n-1}{2}$. Which identity corresponds to the assertion of the problem? Prove this identity directly.]

2. Power series

In the previous Section we have seen examples of how properties of a finite sequence of numbers $(a_0, \dots, a_n)$ can be explored considering the polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$—the generating function of the sequence. But what is the situation if the sequence is infinite—e.g., if it is the sequence of natural numbers or Bernoulli's numbers? Even for an infinite sequence $(a_0, a_1, \dots, a_n, \dots)$ one can write:

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots . \tag{17}$$

But what does the expression on the right-hand side mean? Let us return to the case of finite sequences and polynomials. To deduce properties of various sequences in Sec. 1 we used identities about polynomials. In order to do that it was not necessary to answer such a general questions as "what is a polynomial?", but we had only to know when two polynomials are equal and how to carry out operations with polynomials. We shall answer these questions in connection with the expressions appearing in formula (17) and we shall show then how, using corresponding properties, one can obtain unexpected properties of infinite sequences.

The expression in formula (17) will be called a *power series*. The coefficient a_0 is called the *constant term*. What do we mean by equality of power series? For polynomials, we had two notions of equality which, as was shown in Ch. II, were equivalent. One of them meant that, after cancelling constant terms, the coefficients of the same powers of x were equal. The other notion of equality meant that two polynomials took the same values for the same values of the variable x. The second definition of equality, applied to power series, would need an explanation of what the *value* of a power series for a particular value $x = \alpha$ means. This would then require an explanation of what is the sum of *infinite* number of terms $a_n\alpha^n$. Definitions of this kind can be introduced, although not in every case. But this method is too involved for our purposes. However, if we accept the first notion of equality, then there are no difficult questions. We shall simply say that two power series $f(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$ and $g(x) = b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots$ are equal if $a_0 = b_0$, $a_1 = b_1$, and, generally, $a_n = b_n$ for each n. This definition will be sufficient for our purposes.

If the expansion of a polynomial in powers of x is analogous to the representation of a natural number in the decimal system (we pointed out that analogy at the beginning of Ch. II), then a power series is an analogue of an infinite decimal expansion. This remark was made by Newton.

Let us consider now operations with power series. We shall define them exactly in the same manner as for polynomials—by removing the brackets and collecting like terms. The sum of the power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots$ will be the power series $(a_0 + b_0) + (a_1+b_1)x+(a_2+b_2)x^2+\cdots+(a_n+b_n)x^n+\cdots$. We define the product of these two series by expanding the brackets in the expression $(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)(b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots)$ and collecting like terms. In other words, one has to collect similar terms appearing among expressions $a_nb_mx^{n+m}$. Hence, the coefficient of x^l in the new power series will be the sum $a_0b_l + a_1b_{l-1} + \cdots + a_lb_0$. Note that this is a finite sum, i.e., there will only be *finitely many* similar terms among all the $a_nb_mx^{n+m}$, so that we are able to multiply power series and always obtain a completely determined answer.

We have thus defined operations of addition and multiplication of arbitrary power series. They are defined by the same formulae as for polynomials. Moreover, these operations can also be defined using operations with polynomials themselves. In order to do that, call the polynomial $a_0 + a_1x + \cdots + a_nx^n$, obtained from power series (17) by eliminating all terms with degrees greater than n, the *n-th partial sum* of this series. Note that, for the evaluation of the terms with degrees not exceeding n in the sum or the product of two series, it is necessary to know just the terms with degrees not exceeding n of the two given series. Thus, in order to find the n-th partial sum of the sum or of the product of two series, it is enough to take the n-th partial sums of these series, carry on the respective operation with them (addition or multiplication) and eliminate in the obtained polynomial all terms with degrees exceeding n. Since operations with power series reduce to respective operations with polynomials, they possess the same properties: commutativity, associativity, distributivity. In other words, for power series $f(x)$, $g(x)$, $h(x)$ the following are valid:

$$\begin{gathered} f(x)+g(x)=g(x)+f(x), \\ (f(x)+g(x))+h(x)=f(x)+(g(x)+h(x)), \\ f(x)g(x)=g(x)f(x), \\ (f(x)g(x))h(x)=f(x)(g(x)h(x)), \\ (f(x)+g(x))h(x)=h(x)(f(x)+g(x))=f(x)h(x)+g(x)h(x). \end{gathered}$$

All this long explanation was necessary in order to operate freely with power series, in the same way as with polynomials. This was exactly the point of view of mathematicians of the XVIII century, particularly Euler, who thought of a power series as a polynomial whose degree appeared to be infinite, but the main properties were unchanged. That is the reason we include the Chapter about power series into the theme concerning polynomials ("polynomials with infinite degree").

We can now pass to properties of power series. We shall see that some of the operations with power series can be realised even when the corresponding operations with polynomials cannot.

Theorem 1. *An arbitrary power series $f(x)=a_0+a_1x+\cdots$, such that the constant term a_0 is different from 0, has the inverse power series $f(x)^{-1}$.*

In order to prove the Theorem, we have to find a power series $g(x)=b_0+b_1x+\cdots$, such that $f(x)g(x)=1$. Multiplying power series $f(x)g(x)$ using the rule discussed above, we obtain the power series $a_0b_0+(a_1b_0+a_1b_0)x+\cdots+(a_0b_n+a_1b_{n-1}+\cdots+a_nb_0)x^n+\cdots$. Since this series should to be equal to 1, it is necessary that $a_0b_0=1$, and that the remaining coefficients are equal to 0. Thus, we obtain the equation $a_0b_0=1$ and $b_0=a_0^{-1}$; a_0^{-1} exists since $a_0\neq 0$ by the assumption. The next equation (coefficient of x) gives $a_0b_1+b_0a_1=0$, wherefrom $b_1=-a_0^{-1}b_0a_1=-a_0^{-2}a_1$. In the same way we can recurrently determine the coefficients b_2, b_3, ... from the following equations. Suppose that, considering the coefficients of 1, x, x^2, ... , x^{n-1}, we have already determined b_0, b_1, ... , b_{n-1}. Equating the coefficient of x of $f(x)g(x)$ to 0 gives the equation

$$a_0b_n+a_1b_{n-1}+\cdots+a_nb_0=0,$$

wherefrom $b_n=-a_0^{-1}(a_1b_{n-1}+\cdots+a_nb_0)$. Since b_0, b_1, ... , b_{n-1} have already been determined, this gives us the value of b_n. The Theorem is proved.

In particular, we can see that each polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with the constant term different from 0 has the inverse power series $f(x)^{-1}$. Hence, each rational expression $\dfrac{g(x)}{f(x)}$, where $f(x)$ and $g(x)$ are polynomials and the constant term of the polynomial $f(x)$ is different from 0, can be represented as a power series.

We shall check our conclusion on the simplest possible example. The polynomial $1 - x$ should have the inverse power series $(1 - x)^{-1}$. Let us prove that this series coincides with the series $1 + x + x^2 + x^3 + \cdots$,which has all the coefficients equal to 1. We have to prove that $(1 - x)(1 + x + x^2 + x^3 + \cdots) = 1$. The left-hand side of the equality is equal to $1 + x + x^2 + x^3 + \cdots - x(1 + x + x^2 + x^3 + \cdots)$. We see that all the terms except 1 cancel out. The equality that we have obtained can be written as

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots . \tag{18}$$

Replacing x by $-x$, we obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots . \tag{19}$$

Recall that in Section 3 of Chapter II we corresponded to each sequence $a = (a_0, a_1, a_2, \dots)$ two new sequences: $Sa = (b_0, b_1, \dots)$ and $\Delta a = (c_0, c_1, \dots)$, where $b_0 = a_0$, $b_1 = a_0 + a_1$, $b_2 = a_0 + a_1 + a_2$, $\dots$; $c_0 = a_0$, $c_1 = a_1 - a_0$, $c_2 = a_2 - a_1$, $\dots$ The power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ will be called the *generating function of the sequence* a, by analogy with finite sequences in Sec. 1. How can the generating functions of the sequences Sa and Δa be found? Denote the power series $1+x+x^2+\cdots+x^n+\cdots$ by $s(x)$. The coefficients of the power series $s(x)f(x)$ are in fact equal to a_0, $a_0 + a_1$, $a_0 + a_1 + a_2$, $\dots$, i.e., it is the generating function of the sequence Sa. It is even more obvious that the coefficients of the power series $(1 - x)f(x)$ are equal to a_0, $a_1 - a_0$, $a_2 - a_1$, $\dots$, i.e., it is the generating function of the sequence Δa. Since $s(x) = (1 - x)^{-1}$, the operations of multiplying a power series by $s(x)$ and by $1 - x$ are mutually inverse. It makes the property proved in Sec. 3, Ch. II—that the operations S and Δ are mutually inverse—visually obvious.

We shall pass now to another operation with power series.

Theorem 2. *If the power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ has the constant term different from 0 which has a root of order k, then the whole series $f(x)$ has a root of order k in the form of a power series. This series is uniquely determined by its constant term which can take any value of $\sqrt[k]{a_0}$.*

The Theorem states that under the given assumptions there exists a power series $b_0 + b_1x + b_2x^2 + \cdots$, such that $b_0^k = a_0$ and

$$a_0 + a_1x + a_2x^2 + \cdots = (b_0 + b_1x + b_2x^2 + \cdots)^k. \tag{20}$$

We shall prove this by determining successively the coefficients $b_0, b_1, b_2, \dots$ so that on both sides of relation (20) the terms of degree 0, 1, 2, etc, coincide. Comparing the terms of degree 0, we obtain for b_0 the condition $b_0^k = a_0$. The existence of this number is guaranteed by the assumption. Note also that, since $a_0 \neq 0$, we have $b_0 \neq 0$.

Compare the terms of degree 1. On the right-hand side we can eliminate all the terms of degree greater than 1: b_2x^2 etc., since their multiplication cannot give a term of degree 1. Therefore, the term of degree 1 will be the same as in $(b_0+b_1x)^k$. If we use the binomial formula, we see that the term of degree 1 will be $kb_0^{k-1}b_1x$. The equality of the terms of degree 1 in relation (20) gives $a_1 = kb_0^{k-1}b_1$. Since we have already determined b_0 and it is different from 0, we deduce that $b_1 = \frac{1}{k}b_0^{-(k-1)}a_1$. The terms of degrees 0 and 1 in relation (20) will coincide for these values of b_0 and b_1.

Obviously, we can continue in the same manner. Suppose that coefficients b_0, $b_1, \dots, b_n$ have already been determined so that the terms of degrees $0, 1, \dots, n$ in relation (20) coincide. Let us prove that b_{n+1} can be chosen so that the terms of degree $n+1$ in (20) coincide. Denote by $u(x)$ the n-th partial sum of the power series $f(x)$, i.e., the polynomial $b_0 + b_1x + \cdots + b_nx^n$, and by $v(x)$ the power series $b_{n+2}x^{n+2} + \cdots$. Then the right-hand side of equality (20) takes the form $(u(x) + b_{n+1}x^{n+1} + v(x))^k$. The power series $v(x)$ contains only terms of degree greater than $n+1$, so their multiplication cannot produce terms of degree $n+1$. Therefore, this summand can be eliminated: the terms of degree $n+1$ on the right-hand side of (20) will be the same as in the polynomial $(u(x) + b_{n+1}x^{n+1})^k$. Multiplying out the last expression using the binomial formula, we see that terms of degree $n+1$ can appear only from the summands $u(x)^k + ku(x)^{k-1}b_{n+1}x^{n+1}$. The term of degree $n+1$ entering the polynomial $u(x)^k$ depends only on coefficients of this polynomial, which are already known. Denote this term by $F(b_0, b_1, \dots, b_n)x^{n+1}$. The term of degree $n+1$ in the polynomial $ku(x)^{k-1}b_{n+1}x^{n+1}$ comes from the constant term of polynomial $u(x)$, and so it has the form $kb_0^{k-1}b_{n+1}$. Thus, the term of degree $n+1$ on the right-hand side of equation (20) has the form $(F(b_0, b_1, \dots, b_n) + kb_0^{k-1}b_{n+1})x^{n+1}$. The equality of terms with degree $n+1$ in (20) means that

$$a_{n+1} = F(b_0, b_1, \dots, b_n) + kb_0^{k-1}b_{n+1}.$$

This relation is satisfied if $b_{n+1} = -\frac{1}{k}b_0^{-(k-1)}(a_{n+1} - F(b_0, b_1, \dots, b_n))$. Thus, successively determining coefficients b_n, we can satisfy equality (20). The Theorem is proved.

For example, if $a_0 > 0$, then for the power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ there exists, according to Theorem 2, a unique power series $\sqrt[k]{f(x)}$ with a positive constant term, which can be written down as $f(x)^{1/k}$. Raising it to an arbitrary power m, we obtain the power series $f(x)^{m/k}$, i.e., $f(x)^\alpha$, where α is an arbitrary positive rational number. Applying Theorem 1, we can also write down $f(x)^{-\alpha}$ in the form of power series, and so a power series $f(x)^\alpha$ exists for each rational number α—positive or negative. Some intriguing questions can be posed in connection with this. For example: how can the power series for $(1+x)^\alpha$, α rational, be found explicitly? In other words, can the binomial formula be extended to rational exponents? We derived in Sec. 3, Ch. II the binomial formula for integer exponents, using properties of the derivative of a polynomial. In order to make analogous reasoning in this case, it is necessary to introduce the notion of the derivative of a power series.

We have at our disposal an explicit formula for the derivative of a polynomial (formula (15), Ch. II), which can also be applied to power series. So, for the power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$, we shall define its derivative as the power series

$$f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots . \tag{21}$$

Note that the power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ is equal to the sum of the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ of the degree not exceeding n (its n-th partial sum), and the power series $u(x) = a_{n+1}x^{n+1} + \cdots$, containing terms just of degrees greater than n. Formula (21) shows that the terms of degree less than n in the derivative will be the same as in polynomial $p'(x)$. In other words, $f'(x) = p'(x) + v(x)$, where $v(x)$ is a power series containing only terms with degrees greater than n. That is, the $(n-1)$-st partial sum of the series $f'(x)$ is equal to the derivative of the n-th partial sum of the series $f(x)$. This rule shows which terms of degree less than n are contained in the derivative. Since it is valid for each n, it determines the derivative uniquely.

Using the given rule, one can easily show that the properties of derivative, proved in Sec. 2, Ch. II for polynomials, are valid for power series as well. These are the relations

$$\begin{gathered} (f_1+f_2)' = f_1' + f_2', \qquad (f_1+\cdots+f_n)' = f_1' + \cdots + f_n', \\ (f_1f_2)' = f_1'f_2 + f_1f_2', \qquad (f_1\cdots f_k)' = f_1'f_2\cdots f_k + \cdots + f_1f_2\cdots f_k', \\ (f^k)' = kf^{k-1}f'. \end{gathered} \tag{22}$$

Let us show, for example, how the properties of the derivative of a product can be deduced. Represent each of the power series f_1 and f_2 as the sum of its n-th partial sum and the series containing only terms of degrees greater than n: $f_1 = p_1 + u_1$, $f_2 = p_2 + u_2$. Then $f_1f_2 = p_1p_2 + (p_1u_2 + p_2u_1 + u_1u_2) = p_1p_2 + v$, where v contains only terms of degrees greater than n. Hence, the n-th partial sum p of the series f_1f_2 can be obtained from p_1p_2, eliminating the terms of degrees greater than n, i.e., $p = p_1p_2 + w$, where w is a polynomial containing only terms of degrees greater than n. Using the given rule, we deduce that the $(n-1)$-st partial sum of the series $(f_1f_2)'$ is equal to $p' = (p_1p_2)' + w' = p_1'p_2 + p_1p_2' + w'$ (here we have also used formulae b) and c) from Sec. 2, Ch. II for derivatives of polynomials). Hence, the $(n-1)$-st partial sum of the series $(f_1f_2)'$ is obtained from the polynomial $p_1'p_2 + p_1p_2'$ by eliminating the terms of degrees greater than $n-1$. On the other hand, $f_1' = p_1' + u_1'$, $f_2' = p_2' + u_2'$, $f_1'f_2 + f_1f_2' = p_1'p_2 + p_1p_2' + \varphi$, where $\varphi = p_1'u_2 + p_2u_1' + u_1'u_2 + p_2'u_1 + p_1u_2' + u_1u_2'$ contains only terms of degrees not less than n (even $2n-3$). Thus, the $(n-1)$-st partial sum of the series $f_1'f_2 + f_1f_2'$ is also obtained from the polynomial $p_1'p_2 + p_1p_2'$ by eliminating the terms of degrees greater than $n-1$, and hence it coincides with the $(n-1)$-st partial sum of the series $(f_1f_2)'$. Since this is valid for each n, it follows that $(f_1f_2)' = f_1'f_2 + f_1f_2'$.

The remaining formulae (22) concerning derivatives of products and powers are obtained from the one we have proved, by induction, similarly as for polynomials.

Formulae for the derivatives of sums are completely obvious—we leave their proofs to the reader.

We are now ready to deduce the formula for the power series for $(1+x)^\alpha$, where α is a rational number. We shall consider only the case of positive α. Let $\alpha = p/q$, where p and q are natural numbers. According to Theorem 2, there exists a power series $f(x) = 1 + a_1x + \cdots$, such that

$$f(x)^q = (1+x)^p. \tag{23}$$

This series will be denoted by $f(x)^{p/q}$. Consider the derivatives of both sides of equality (23). Using the properties (22) of derivatives of power series that we have found, and corresponding properties for polynomials ((17), Ch. II), we obtain

$$qf'(x)f(x)^{q-1} = p(1+x)^{p-1}.$$

Multiplying both sides of this equality by $(1+x)f(x)$, we obtain

$$qf'(x)f(x)^q(1+x) = p(1+x)^p f(x).$$

Using now equality (23), we can divide the left-hand side by $f(x)^q$ and the right-hand side by $(1+x)^p$. Recalling that $p/q = \alpha$, we obtain

$$f'(x)(1+x) = \alpha f(x). \tag{24}$$

Let $f(x) = 1 + a_1x + \cdots + a_nx^n + \cdots$. Equating coefficients of x^{n-1} on both sides of equality (24) and taking into account that, by definition (21), $f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots$, we obtain that

$$nan + (n-1)a_{n-1} = \alpha a_{n-1},$$

wherefrom $a_n = \dfrac{\alpha - n + 1}{n} a_{n-1}$. Using this formula several times, we obtain

$$a_n = \frac{(\alpha-n+1)(\alpha-n+2)\cdots(\alpha-n+r)}{n(n-1)\cdots(n-r+1)} a_{n-r}.$$

Since $a_0 = 1$, for $r = n$ we get $a_n = \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. In other words,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots. \tag{25}$$

Note that if $\alpha = m$ is an integer, all coefficients of this series starting from the $(m+1)$-st vanish, and we obtain the usual binomial formula. Formula (25) is valid for negative α, too (Problem 7). In fact, formula (25) should be called *Newton's binomial formula*, since it was Newton who derived it (even for real exponents α); the formula for natural exponents was known much earlier, e.g., to Pascal.

As a conclusion of this Section we consider an application of the generalized binomial formula (25) to the so called *Catalan numbers*. They are connected with various problems of partitions of numbers. Suppose, for example, that we wish to calculate the product of n numbers $a_1, a_2, \ldots, a_n$, taken in a prescribed order, but we want to find that product by using successive multiplications of just two numbers at a time. In order to do that, we have to put brackets into the product $a_1 \cdots a_n$ so that in each bracket there are always two factors, which may themselves be expressions in brackets. The number of ways that the brackets can be arranged is the Catalan number c_n. We shall put $c_1 = 1$. Obviously, $c_2 = 1$. For a product of three numbers $a_1a_2a_3$, two ways of inserting brackets are possible: $(a_1a_2)a_3$ and $a_1(a_2a_3)$, so that $c_3 = 2$. For $n = 4$ the possibilities are $((a_1a_2)a_3)a_4$, $(a_1a_2)(a_3a_4)$ and $a_1(a_2(a_3a_4))$, so that $c_4 = 3$.

Catalan numbers satisfy an important relation. The last product that we find when calculating $a_1 \cdots a_n$ determines an arrangement of brackets $(a_1 \cdots a_k) \times (a_{k+1} \cdots a_n)$. Inside each of the brackets we can arrange other brackets in an arbitrary manner, i.e., in c_k ways in the first, and in c_{n-k} ways in the second one. In total, there will be c_kc_{n-k} arrangements. The total number of all arrangements is equal to the sum of all these numbers for $k = 1, 2, \ldots, n-1$. In other words, the following relation is valid

$$c_n = c_1c_{n-1} + c_2c_{n-2} + \cdots + c_{n-1}c_n, \qquad \text{for } n \geqslant 2. \tag{26}$$

The right-hand side of relation (26) looks like the formula for coefficients of the product of two power series, and so it suggests to consider the power series

$$f(x) = c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

(the generating function for Catalan numbers). The right-hand side in (26) is equal to the coefficient of x^n in the series $(f(x))^2$. Relation (26) shows that the coefficients of the series $f(x)$ and $(f(x))^2$ will be the same for all terms of degree 2 and higher. But, $f(x)$ has the term x of degree 1, while $(f(x))^2$ has no such term. Hence, $(f(x))^2 = f(x) - x$. Thus our series satisfies the quadratic equation $y^2 - y + x = 0$ and can be found explicitly:

$$f(x) = \frac{1}{2}(1 - \sqrt{1-4x}).$$

We take the "minus" sign because the series $\sqrt{1-4x}$ has a constant term (1), and $f(x)$ has no such term.

According to formula (25),

$$\sqrt{1-4x} = 1 + \frac{1}{2}(-4x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(-4x)^2 + \cdots + \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4x)^n + \cdots .$$

Hence, $c_n = -\frac{1}{2}\frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4)^n$. The formula can be simplified further:

$$c_n = -\frac{1}{2}\frac{(-1)(-3)\cdots(-2n+3)}{n!}(-2)^n = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-3)}{n!}2^{n-1}.$$

Multiply both numerator and denominator of the last expression by $(n-1)!$ and combine each factor from $1 \cdot 2 \cdots (n-1)$ with one of the factors 2. We obtain the product of all even natural numbers not exceeding $2n-2$. In the numerator there is already the product of odd numbers smaller than $2n-2$. Together, these products give $(2n-2)!$. As a result we obtain

$$c_n = \frac{(2n-2)!}{n!\,(n-1)!}.$$

Since $C_{2n-2}^{n-1} = \dfrac{(2n-2)!}{(n-1)!\,(n-1)!}$, the last formula can also be written as

$$c_n = \frac{1}{n} C_{2n-2}^{n-1}.$$

Problems

1. Find the coefficients of the power series $\dfrac{1}{(1-x)^2}$ by squaring the series $\dfrac{1}{1-x}$.

2. Find a formula for the coefficients of the power series $\dfrac{1}{(1-x)^n}$. [*Hint*: Use induction on n and the connection between multiplying a series by $\dfrac{1}{1-x}$ and applying operation S to its coefficients.]

3. Find the coefficients of the power series $\dfrac{1}{(1-ax)(1-bx)}$.

4. Prove the formula $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$, where f and g are polynomials or power series.

5. Prove that the series $1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$ and $1 - x + \dfrac{x^2}{2!} + \cdots + (-1)^n \dfrac{x^n}{n!} + \cdots$ are mutually inverse.

6. Find the formula for the derivative of $\dfrac{1}{(1-x)^n}$. Use this formula to determine the coefficients of the power series $\dfrac{1}{(1-x)^n}$ (using induction on n).

7. Find all power series $f(x)$ for which $f'(x) = f(x)$.

8. Prove that formula (25) is also valid for negative α. [*Hint*: Put $\alpha = -p/q$, where p and q are natural, $f(x) = (1+x)^\alpha$ and use the relation $f(x)^q(1+x)^p = 1$. Check that for negative integers α the result coincides with the result of Problems 2 and 6.]

9. In how many ways can a convex $(n+1)$-gon be divided into triangles by its diagonals, not intersecting inside the polygon? Prove that this number is equal to the Catalan number c_n.

10. Let $f(x)$ be a polynomial of degree n. Prove that the coefficient of x^k in the power series $\dfrac{f(x)}{1-x}$ is equal to $f(1)$, if $k > n$.

11. Let $f_n(x) = x + 2^n x^2 + 3^n x^3 + \cdots$. Prove that $f_n(x) = \dfrac{u_n(x)}{(1-x)^{n+1}}$, where $u_n(x)$ is a polynomial of degree $n+1$ satisfying the relation $u_{n+1}(x) = x(1-x)u_n'(x) + (n+1)xu_n(x)$. [*Hint*: Prove that $xf_n'(x) = f_{n+1}(x)$. Find $f_0(x)$ and use induction.]

12. Prove that

$$n^n - C_n^1(n-1)^n + C_n^2(n-2)^n + \cdots + (-1)^{n-1}C_n^{n-1} \cdot 1 = n!.$$

[*Hint*: Use Problems 10 and 11. Prove that in Problem 11, $u_n(i) = n!$.]

3. Partitio Numenorum

The Latin term *partitio numenorum*—partition of numbers—is the name given by Euler to the part of Mathematics investigating partitions of natural numbers using power series. As an introduction, we gave in Sec. 1 examples of problems with partitions which can be solved using polynomials.

For more general cases we need infinite sums of power series. Let $f_n(x)$, $n = 0, 1, 2, \ldots$, be an infinite sequence of power series, such that each series $f_n(x)$ starts with a certain power of x, which increases when n increases. In other words, for each exponent N, the term ax^N will be distinct from 0 for only finitely many series $f_n(x)$. Then, when evaluating the coefficient of x^N in the infinite sum $f_0(x)+f_1(x)+\cdots+f_n(x)+\cdots$ we have to sum up only a finite number of series: $f_0(x)+f_1(x)+\cdots+f_m(x)$. Hence, the whole N-th partial sum of the resulting series will coincide with the N-th partial sum of the finite sum of series $f_0(x)+f_1(x)+\cdots+f_m(x)$. Because of that, the evaluation of the infinite sum (i.e., its partial sums) always reduces to the evaluation of partial sums of certain finite sums of series. Thus the rules that we deduced in Sec. 2 for finite sums of series are valid also for infinite sums (if the series $f_n(x)$ satisfy the formulated condition). As a matter of fact, only after these explanations can we say that a power series $f(x)$ is the sum of its terms—in this case $f_n(x) = a_n x^n$.

The same remarks apply also for infinite products of the form

$$(1 + f_0(x))(1 + f_1(x))(1 + f_2(x)) \cdots (1 + f_n(x)) \cdots, \tag{27}$$

where the power series $f_n(x)$ satisfy the same condition: $f_n(x)$ starts with a power of x which increases unboundedly when n increases. Then for each exponent N, the series $f_k(x)$ do not contain terms with degree N, starting from some number $m+1$, i.e., for $k > m$. Therefore, the terms with a fixed exponent N in the product (27) are obtained from the finite product $(1 + f_0(x)) \cdots (1 + f_m(x))$.

Using these observations, we can now find the generating functions for numbers of partitions of various kinds.

For example, the numbers of partitions to summands not exceeding m, have the generating function

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) \cdots (1 + x^m + x^{2m} + x^{3m} + \cdots).$$

In this partition of number n, the number 1 appears a_1 times, number 2 a_2 times, ..., number m a_m times: $n = 1 \cdot a_1 + 2 \cdot a_2 + \cdots + m \cdot a_m$ (some of the numbers a_i may be equal to 0). To this partition there corresponds the term obtained by multiplying the term x^{a_1} form the first bracket, x^{2a_2} from the second, ..., x^{ma_m} from the m-th brackets. It means that the coefficient of x^n is equal to the total number of all partitions of n into summands not exceeding m. Taking into account formula (18) we can write this series in the form

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^m)}. \tag{28}$$

In a completely analogous way, the number of partitions of number n into arbitrary natural summands has the generating function

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^m)\cdots}. \tag{29}$$

The number of partitions into odd summands has the generating function

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots(1-x^{2m+1})\cdots}, \tag{30}$$

and into even summands:

$$\frac{1}{(1-x^2)(1-x^4)\cdots(1-x^{2m})\cdots}.$$

If we are interested only in partitions into *distinct* summands, then the generating function is

$$(1+x)(1+x^2)(1+x^3)\cdots(1+x^m)\cdots. \tag{31}$$

Here we allow partitions in which 1 appears a_1 times, 2 appears a_2 times, ..., m appears a_m times, but a_i can be only 0 or 1. But the factors of product (31) contain exactly those terms x^{a_1} ($a_1 = 0$ or 1) in the first factor, x^{2a_2} ($a_2 = 0$ or 1) in the second one, etc.

These formulae have several applications.

Theorem 3. *The number of partitions of number n into distinct summands is equal to the number of its partitions into odd summands (some of them possibly equal to each other).*

For example, number 6 has 3 partitions into distinct summands: $6 = 1+5 = 1+2+3 = 2+4$ and also 3 partitions into odd summands: $6 = 1+5 = 1+1+1+3 = 3+3$.

In terms of generating functions, the Theorem means that power series (30) and (31) coincide. In order to prove this, write series (31) in the form

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots.$$

In the numerator there is the product of all factors $1-x^{2n}$, and in the denominator of all $1-x^m$. The factors $1-x^m$ from the denominator with even m give the expression appearing in the numerator and so after cancelling there remains the product of factors $1-x^m$ with odd m, i.e., the series (30).

The next property concerns the number of partitions of a natural number n into arbitrary natural summands. Denote the number of all such partitions by $p(n)$. As we have already seen, the series $1+p(1)x+p(2)x^2+\cdots+p(n)x^n+\cdots$ is given by formula (29).

Theorem 4. *For each $n \geqslant 2$, the following inequality holds*

$$p(n)-2p(n-1)+p(n-2) \geqslant 0.$$

In other words, if we draw the points with coordinates $(n, p(n))$, $n=1,2,\dots$, in the plane, then each point lies below the segment joining the two neighbouring ones, Fig. 1. That is, if we knock small nails into the plane and span a string, we shall obtain a convex infinite polygon. The first 10 values of the sequence $p(n)$ are: $p(1)=1$, $p(2)=2$, $p(3)=3$, $p(4)=5$, $p(5)=7$, $p(6)=11$, $p(7)=15$, $p(8)=22$, $p(9)=30$, $p(10)=42$. You can check the Theorem for these values experimentally. The convexity of the obtained polygon is connected with the fact that numbers $p(n)$ increase very fast: $p(50)=204\,226$.

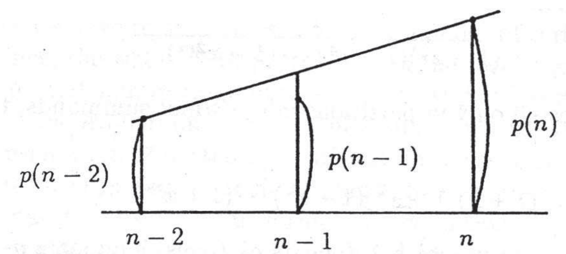


Fig. 1

Before proving the Theorem, we make one remark. We corresponded in Sec. 3, Ch. II to each sequence $a=(a_0,a_1,a_2,\dots)$ the sequence $\Delta a=(a_0,a_1-a_0,a_2-a_1,\dots)$. Apply to the last sequence the same operation once more. We obtain the sequence $b=\Delta\Delta a=(a_0,a_1-2a_0,a_2-2a_1+a_0,\dots)$. The term b_n of this sequence has the form $a_n-2a_{n-1}+a_{n-2}$ for $n \geqslant 2$. On the other hand, if $f(x)=a_0+a_1x+\cdots$ is the generating function of sequence a, then, as we saw in Sec. 2, the generating function of the sequence Δa will be $(1-x)f(x)$. Hence, the sequence $\Delta\Delta a$ has the generating function $(1-x)^2f(x)$ and we arrive at the identity

$$(1-x)^2f(x)=a_0+(a_1-2a_0)x+(a_2-2a_1+a_0)x^2+\cdots+(a_n-2a_{n-1}+a_{n-2})x^n+\cdots$$

if $f(x)=a_0+a_1x+a_2x^2+\cdots$.

We can now proceed to the proof of the Theorem. Taking the last remark into account, it asserts that the coefficients of the series $(1-x)^2(p(0)+p(1)x+$

$\cdots + p(n)x^n + \cdots)$, starting with that of x^2, are nonnegative. Since $p(0) + p(1)x + p(2)x^2 + \cdots = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots$, we have to prove that coefficients of the series $(1-x)(1-x^2)^{-1}(1-x^3)^{-1}\cdots$ are nonnegative starting with the coefficient of x^2.

Put $g(x) = (1-x^3)^{-1}(1-x^4)^{-1}\cdots$. The series we are interested in has the form $(1-x)(1-x^2)^{-1}g(x)$. Since $(1-x)(1-x^2)^{-1} = \dfrac{1-x}{(1-x)(1+x)} = (1+x)^{-1}$, this series is equal to $(1+x)^{-1}g(x)$. Arguing in the same manner as with series (28), (29) and (30), we can convince ourselves that $g(x)$ is the generating function for numbers of partitions into summands not exceeding 3. Denoting by $q(n)$ the number of partitions of n of this kind, we find that $g(x) = 1 + q(1)x + q(2)x^2 + \cdots$. Since $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$, the coefficient of x^n in the series $(1+x)^{-1}g(x)$ is equal to $q(n) - q(n-1) + q(n-2) - \cdots + (-1)^n$ (we only have to remember the rule for multiplication of power series: each term of the first one is multiplied by each term of the second, and then similar terms are reduced). Thus, it remains to prove the inequality

$$q(n) - q(n-1) + q(n-2) - \cdots + (-1)^n \geqslant 0. \tag{32}$$

This follows from the obvious inequality $q(n) \geqslant q(n-1)$: really, enlarging by 1 the largest summand of a certain partition of the number $n-1$, we obtain a partition of number n. If the former consisted of summands greater than 2, then the latter would have the same property. From this inequality, it follows that sum (32) splits into $\frac{n}{2}$ nonnegative differences $q(n-2k) - q(n-2k-1)$ (for n odd) and one extra summand equal to 1 (for n even). This proves the Theorem.

Up to now we have considered properties of numbers of partitions practically starting from nothing—just using multiplication of power series. It was Euler who found a more subtle method for calculating coefficients of some products, using the so called functional equations that these products satisfy. We shall illustrate this method on an example.

Consider the question of partitions of a natural number into a given number of *distinct* summands. In order to do this, Euler proposed to introduce a new variable z and the series $G(x,z) = (1+z)(1+xz)(1+x^2z)(1+x^3z)\cdots$. Expanding this series in powers of z, we obtain the equality

$$G(x,z) = 1 + u_1(x)z + u_2(x)z^2 + \cdots + u_m(x)z^m + \cdots, \tag{33}$$

where $u_i(x)$ is a power series in variable x. Here the term $u_m(x)z^m$ contains the terms obtained by multiplying m terms x^iz from the product $G(x,z)$, and so the coefficient of x^nz^m is equal to the number of partitions of n into m distinct summands. In other words, $u_m(x)$ is exactly the generating function of such partitions.

If we replace z in the product $G(x,z)$ by xz, we obtain all the factors of $G(x,z)$ except the first one. Therefore

$$G(x,z) = (1+z)G(x,xz). \tag{34}$$

This is a functional equation for the series $G(x,z)$. On the other hand, replacing z by xz in the series (33), we see that

$$G(x,xz) = 1 + u_1(x)xz + u_2(x)x^2z^2 + \cdots + u_m(x)x^mz^m + \cdots .$$

Multiplying this expression for $G(x,xz)$ by $1+z$, we obtain that $u_m(x) = u_m(x)x^m + u_{m-1}(x)x^{m-1}$, wherefrom

$$u_m(x) = \frac{x^{m-1}}{1-x^m}u_{m-1}(x). \tag{35}$$

Applying the same relation to $u_{m-1}(x)$ and substituting into equality (35), we obtain that

$$u_m(x) = \frac{x^{(m-1)+(m-2)}}{(1-x^m)(1-x^{m-1})}u_{m-2}(x).$$

Repeating this operation m times and taking into account that $u_0(x) = 1$, we find for $u_m(x)$ the expression

$$u_m(x) = \frac{x^{(m-1)+(m-2)+\cdots+1}}{(1-x^m)(1-x^{m-1})\cdots(1-x)} = \frac{x^{\frac{m(m-1)}{2}}}{(1-x)(1-x^2)\cdots(1-x^m)}. \tag{36}$$

But we have already met the series $\dfrac{1}{(1-x)\cdots(1-x^m)}$ (see formula (28)). This is the generating function for the numbers of partitions into summands not exceeding m. Formula (36) shows that *the number of partitions of number n into m distinct summands is equal to the number of partitions of number $n - \dfrac{m(m-1)}{2}$ into arbitrary summands not exceeding m.*

In connection with the generating function (29) Euler considered a logically easier product

$$(1-x)(1-x^2)(1-x^3)\cdots . \tag{37}$$

This is a very interesting expression. We mentioned in Sec. 1 the analogy between Gauss polynomials $g_{k,l}(x)$ and binomial coefficients. The analogy was based on formula (7) in which the polynomial $h_m(x)$ was analogous to $m!$. Recall that

$$h_m(x) = (1-x)(1-x^2)\cdots(1-x^m).$$

From this point of view, product (37) is analogous to "factorial of infinity". This expression has no meaning for numbers or for polynomials, but it describes a completely determined expression if we use power series.

Euler expanded this expression up to the term x^{51} and obtained the expression

$$\begin{aligned}(1-x)(1-x^2)(1-x^3)\cdots &= \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \cdots\end{aligned}$$

He was shocked by the regularity he noticed: all coefficients are equal to 0, +1 or −1. Moreover, the exponents of terms with nonzero coefficients form a sequence that was known to Euler: these are numbers of the form $\frac{n(3n+1)}{2}$ for $n = -1$, 1, -2, 2, -3, 3, -4, 4, -5, 5, -6. These numbers attracted interest at the time, in connection with the so called "figured numbers", known from ancient times. For example, a triangular number is the number of dots in a regular triangle with $n+1$ equally spaced dots along each side and height n (Fig. 2). Thus, these are the numbers: 1, 3, 6, 10, ... , $\frac{n(n+1)}{2}$, ... A square number is a number of equally spaced dots in a square of side n (Fig. 3). In other words, these are simply exact squares: n^2.

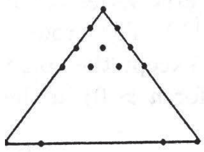


Fig. 2

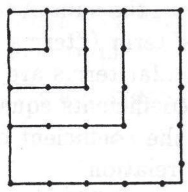


Fig. 3

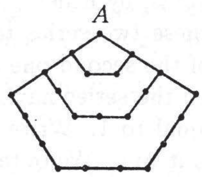


Fig. 4

Pentagonal numbers are obtained starting from a regular pentagon. Put a dot in one vertex, call it A, and add four dots to create a regular pentagon of side 1. Then extend the two sides starting from A and complete a regular pentagon of side 2. The n-th pentagonal number is equal to the number of dots after the completion of the regular pentagon of side $n-1$, Fig. 4. Consequently, the n-th pentagonal number is equal to the sum of the arithmetic progression: $1+4+7+\cdots+(3n-2) = \frac{3n^2-n}{2}$. By analogy, the numbers of the same kind but with $n=-m$, i.e. the numbers $\frac{3m^2+m}{2}$, are also called pentagonal numbers.

Euler proposed for the product $(1-x)(1-x^2)(1-x^3)\cdots$ an expression in the form of a power series with terms $(-1)^n x^{\frac{n(3n-1)}{2}} + (-1)^n x^{\frac{n(3n+1)}{2}}$ for $n = 0, 1, 2, \ldots$. He referred to this as "an important observation which at this time I cannot prove with geometric rigor". We call such an observation a hypothesis. This hypothesis was stated by Euler in 1741. He found the proof nine years later, in 1750. Because of its connection with pentagonal numbers, it is now called Euler's Pentagonal Theorem. Its proof is a bit more involved than arguments we have seen so far, so we postpone it to the Appendix.

Euler's Pentagonal Theorem gives new properties of numbers of partitions. First of all, the product $(1-x)(1-x^2)(1-x^3)\cdots$ is also a generating function. Namely, analogously to expanding the product (31), each term x^n is obtained from some partition of n into distinct summands. But now this term enters with the sign "+" if the number of summands is even, and with the sign "−" if this number is odd. Thus, the coefficient of x^n is equal to the *difference* between the number of partitions of n into even and odd number of distinct summands. Therefore, the Pentagonal Theorem states:

The number of partitions of a natural number n into an even number of distinct summands is equal to the number of its partitions into an odd number of distinct summands, unless n is a pentagonal number, in which case the difference between these two numbers is equal to $(-1)^m$ if n has the form $\dfrac{m(3m-1)}{2}$ for a positive or negative integer m.

Another corollary of the Pentagonal Theorem is the following. Recall that product (29) coincides with the series $1 + p(1)x + p(2)x^2 + \cdots$, and the inverse product (37) is, by Euler's Theorem, the sum of terms of the form $(-1)^n x^{\frac{n(3n\pm1)}{2}}$. Multiply these two series term by term ("terms of the first series are multiplied by terms of the second one and similar terms are reduced"). The product is equal to 1, i.e., to the series having all coefficients equal to 0, except the constant term which is equal to 1. Write down the coefficient of x^n (for $n > 0$) in the product and equate it to 0. We obtain the relation

$$p(n) - p(n-1) - p(n-2) + p(n-5) + \cdots = 0.$$

The terms of this sum are $(-1)^n(p(n-m_1)+p(n-m_2))$, where $m_1 = \dfrac{m(3m-1)}{2}$, $m_2 = \dfrac{m(3m+1)}{2}$, including only those values of m_1 and m_2 that do not exceed n, and $p(0)$ is taken equal to 1. The relation expresses $p(n)$ in terms of $p(n')$ with values $n' < n$ and it gives a suitable method of evaluating the numbers $p(n)$ recurrently. For instance,

$$\begin{aligned}
p(10) &= p(9) + p(8) - p(5) - p(3),\\
p(9) &= p(8) + p(7) - p(4) - p(2),\\
p(8) &= p(7) + p(6) - p(3) - p(1),\\
p(7) &= p(6) + p(5) - p(2) - 1,\\
p(6) &= p(5) + p(4) - p(1),\\
p(5) &= p(4) + p(3) - 1,\\
p(4) &= p(3) + p(2),\\
p(3) &= p(2) + p(1),\\
p(2) &= p(1) + 1,\\
p(1) &= 1,
\end{aligned}$$

wherefrom, going up from the bottom line, we obtain $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$, $p(7) = 15$, $p(8) = 22$, $p(9) = 30$, $p(10) = 42$.

Problems

1. Denote by a_n the number of ways in which the sum of n cents can be formed using 1, 5, 10 and 50 cent coins. Prove that the power series $1 + a_1x + a_2x^2 + \cdots$ is equal to $\dfrac{1}{p(x)}$, where p is a polynomial, and find this polynomial.

2. Do the same for the number a_n of ways in which a number n can be decomposed into summands equal to given numbers $k_1, \ldots, k_r$.

3. Prove that if in Problem 2 partitions that differ in the order of summands are considered different, then the number of partitions of n into m summands is equal to the coefficient of x^n in $(x^{k_1} + \cdots + x^{k_r})^m$. Solve Problem 2 under these new conditions.

4. Prove that each natural number n can be represented in 2^{n-1} ways as a sum of natural summands, if partitions that differ in the order of summands are considered different.

5. How many distinct monomials of degree m in n variables $x_1, \ldots, x_n$ are there? [*Hint*: Represent the sum of all the monomials $x_1^{r_1} \cdots x_n^{r_n}$ in the form $\dfrac{1}{p(x_1, \ldots, x_n)}$, where $p(x_1, \ldots, x_n)$ is a polynomial, and then put $x_1 = \cdots = x_n = y$.]

6. Put $F_m = \dfrac{1}{(1-x)(1-x^2)\cdots(1-x^m)}$. The obvious relation $F_m(1-x^m) = F_{m-1}$ implies the equality $F_m = F_{m-1} + x^m F_m$. Deduce that the number of partitions of a number n into summands $1, \ldots, m$ is equal to the sum of numbers of such partitions of number $n-m$ and the number of partitions of n to summands $1, 2, \ldots, m-1$. What relation holds for $n = m$?

Note that the number of partitions of n into summands $1, 2, \ldots, m$ is equal to $P_{n,m}(n)$ in the notation of Sec. 1. Hence, the relation obtained is a consequence of equality (2) of Sec. 1 (for which values of k, l, n?). But this time it is obtained without any arguments about partitions, but using exclusively properties of power series.

7. Euler was attracted to the theory of partitions by a German mathematician called Node. In a letter Node asked: how can the number of partitions for an arbitrary large number n be determined? For example, what is the number of partitions of 50 into summands not exceeding 7? Or into 7 summands? Euler answered Node's question in 2 weeks, showing the connection of this problem with power series. He published his results half a year later. In particular, he deduced the relation mentioned in Problem 6, and found the sequence of numbers of partitions, starting from small n and m, up to $n = 69$ and $m = 11$.

Try to reconstruct Euler's argument and make a table for numbers of partitions of $1, 2, \ldots, 49, 50$ into summands not exceeding $1, 2, \ldots, 7$. Prove that the number of partitions of 50 into summands not exceeding 6 is equal to 18138, and that the number of partitions of 50 into 7 distinct summands is 522. [*Hint*: Use also a relation following from formula (36).]

8. Prove that the number of partitions of n in which only odd summands can be equal to each other is equal to the number of partitions in which each summand appears at most 3 times. [*Hint*: Represent both numbers of partitions using generating functions which are expanded into infinite products of prime factors.]

9. Represent the product $(1+xz)(1+x^2z)(1+x^2z)(1+x^8z)\cdots$ as a series $1+u_1(x)z+u_2(x)z^2+\cdots$ and find $u_k(x)$.

10. Represent the power series $(1-xz)^{-1}(1-x^2z)^{-1}(1-x^3z)^{-1}\cdots$ in the form $1+v_1(x)z+v_2(x)z^2+\cdots$ and find $v_k(x)$. [*Hint*: Use a functional equation for this series.]

Use the obtained result to deduce a relation between numbers of partitions, as it was done with formula (36).

APPENDIX I

Euler's Pentagonal Theorem

There are several different proofs of Euler's theorem, and he we shall present two of them. We start with the proof derived by Euler himself. It is a remarkable piece of pure algebra. It does not use anything but multiplying brackets and grouping the terms, but these operations have to be so finely combined that Euler himself found the proof nearly ten years after he formulated the Theorem as a hypothesis.

The idea of the proof is very natural. We shall expand, step by step, the product

$$(1-x)(1-x^2)(1-x^3)\cdots(1-x^n)\cdots, \tag{1}$$

representing it in each step as a sum of a polynomial of degree N and an expression divisible by x^{N+1}, where N increases with each step. Thus, we shall evaluate partial sums of the expansion of product (1) into a power series.

Let us start with a finite product $(1-a_1)(1-a_2)\cdots(1-a_n)$. Removing the last bracket, we shall write it in the form

$$(1-a_1)(1-a_2)\cdots(1-a_n) = \\ = (1-a_1)(1-a_2)\cdots(1-a_{n-1}) - a_n(1-a_1)(1-a_2)\cdots(1-a_{n-1}).$$

Now the same method can be applied to the first summand, and it divides into the product of $n-2$ factors $(1-a_1)(1-a_2)\cdots(1-a_{n-2})$ and the term $-a_{n-1}\times(1-a_1)(1-a_2)\cdots(1-a_{n-2})$. Next, the term $(1-a_1)(1-a_2)\cdots(1-a_{n-2})$ is transformed in the same manner, and we can produce $n-1$ such transformations, until we finish with the term $(1-a_1)$. As a result we obtain the identity

$$(1-a_1)(1-a_2)\cdots(1-a_n) = 1-a_1-a_2(1-a_1)- \\ -a_3(1-a_1)(1-a_2)-\cdots-a_n(1-a_1)(1-a_2)\cdots(1-a_{n-1}). \tag{2}$$

Identity (2) can be applied to the infinite product $(1-u_1)\cdots(1-u_n)\cdots$, where $u_i(x)$ are power series, starting with larger and larger degrees of x: we shall apply such reasoning to the case when $u_n = x^n$, so that the "series" starts with the term of degree n (and finishes with it). We obtain the identity

$$(1-u_1)(1-u_2)\cdots(1-u_n)\cdots = 1-u_1-u_2(1-u_1)-\cdots-u_n(1-u_1)\cdots(1-u_{n-1})-\cdots. \tag{3}$$

Really, if we consider the terms of degree not exceeding n, then on the left-hand side we can eliminate all factors starting from the $(n+1)$-st, and on the right-hand side all the summands starting from the $(n+1)$-st, since they do not contain terms of degree not exceeding n. But then we obtain identity (2) with $a_i = u_i$. That is, the terms of degree not exceeding n on the left and on the right coincide. Since it holds for each n, identity (3) is true.

Substituting in relation (3) $u_i = x^i$, we obtain the equality

$$(4)\quad (1-x)(1-x^2)\cdots(1-x^n) = 1 - x - x^2(1-x) - \\ - x^3(1-x)(1-x^2) - \cdots - x^n(1-x)\cdots(1-x^{n-1}) - \cdots .$$

This is the first step in our chain of transformations. Denote the product $(1-x)\cdots(1-x^n)\cdots$ by P_0, take out x^2 from all the terms except the first two in equality (4), and put

$$P_1 = 1 - x + x(1-x)(1-x^2) + \cdots + x^m(1-x)\cdots(1-x^{m+1}) + \cdots .$$

Then the equality (4) takes the form

$$(5)\qquad P_0 = 1 - x - x^2 P_1.$$

Transform now P_1. Write it in the form

$$P_1 = Q_0 + Q_1 + \ldots Q_k + \cdots ,$$

where $Q_k = x^k(1-x)(1-x^2)\cdots(1-x^{k+1})$. Remove the first bracket in the product Q_k. We obtain the equality which can be written as $Q_k = A_k - B_k$, where

$$A_k = x^k(1-x^2)(1-x^3)\cdots(1-x^k), \qquad A_0 = 1,$$
$$B_k = x^{k+1}(1-x^2)(1-x^3)\cdots(1-x^{k+1}), \qquad B_0 = x.$$

Expression P_1 can be written as

$$(6)\qquad P_1 = A_0 - B_0 + A_1 - B_1 + \cdots + A_k - B_k + \cdots .$$

Note now that, for $k \geqslant 2$, the expression $A_k - B_{k-1}$ can be written more simply:

$$A_k - B_{k-1} = x^k(1-x^2)\cdots(1-x^{k+1}) - x^k(1-x^2)\cdots(1-x^k) = -x^{2k+1}C_{k-2},$$

where $C_k = (1-x^2)\cdots(1-x^{k+2})$, $k \geqslant 2$, $C_0 = 1 - x^2$, and so

$$C_{k-2} = (1-x^2)(1-x^3)\cdots(1-x^k).$$

Writing the expansion (6) in the form

$$P_1 = A_0 - B_0 + A_1 + (-B_1 + A_2) + (-B_2 + A_3) + \cdots + (-B_{k-1} + A_k) + \cdots ,$$

we obtain the representation $P_1 = 1 - x + x(1-x^2) - x^5 C_0 - x^7 C_1 - \cdots - x^{2k+5} C_k - \cdots$. This can also be written as

$$(7)\qquad P_1 = 1 - x^3 - x^5 P_2,$$

where $P_2 = C_0 + x^2C_1 + \cdots + x^{2k}C_k + \cdots$, or, in expanded form,

$$P_2 = 1 - x^2 + x^2(1-x^2)(1-x^3) + \cdots + x^{2k}(1-x^2)(1-x^3)\cdots(1-x^{k+2}) + \cdots .$$

The main part of the proof is now finished. Equalities (5) and (7) give us a basis for inductively expanding product (1) and finding partial sums with greater and greater degrees. It remains to formulate the process of passing from the n-th step to the $(n+1)$-st and to write down the result.

Put

$$\begin{aligned} P_n = 1 - x^n + x^n(1-x^n)(1-x^{n+1}) + x^{2n}(1-x^n)(1-x^{n+1})(1-x^{n+2}) \\ + \cdots + x^{kn}(1-x^n)(1-x^{n+1})\cdots(1-x^{n+k}) + \cdots . \end{aligned}$$

Transform this expression in the same way as we have done with P_1. Put $x^{kn}(1-x^n)(1-x^{n+1})\cdots(1-x^{n+k}) = Q_k$. Then $P_0 = Q_0 + Q_1 + \cdots + Q_k + \cdots$. Expand in the product Q_k the first bracket: $Q_k = A_k - B_k$,

$$A_k = x^{nk}(1-x^{n+1})\cdots(1-x^{n+k}), \qquad B_k = x^{n(k+1)}(1-x^{n+1})\cdots(1-x^{n+k}).$$

Consider the difference $A_k - B_{k-1}$ for $k \geqslant 2$:

$$\begin{aligned} A_k - B_{k-1} &= x^{nk}(1-x^{n+1})\cdots(1-x^{n+k-1})(-x^{n+k}) \\ &= -x^{nk+n+k}(1-x^{n+1})\cdots(1-x^{n+k-1}) = (-x^{nk+n+k})C_{k-2}, \end{aligned}$$

where $C_k = (1-x^{n+1})\cdots(1-x^{n+1+k})$, $k \geqslant 0$. The exponent $nk+n+k$ of the power of x that divides C_{k-2} can be written in the form $nk+n+k = (n+1)(k-2)+3n+2$. Hence, P_n can be written as

$$\begin{aligned} P_n &= A_0 - B_0 + A_1 + (-B_1 + A_2) + \cdots + (-B_{k-1} + A_k) + \cdots \\ &= A_0 - B_0 + A_1 + (-x^{3n+2})(C_0 + x^{n+1}C_1 + x^{2(n+1)}C_2 + \cdots). \end{aligned}$$

The sum

$$\begin{aligned} &C_0 + C_1 + C_2 + \cdots = \\ &= 1 - x^{n+1} + x^{n+1}(1-x^{n+2}) + \cdots + x^{k(n+1)}(1-x^{n+2})(1-x^{n+3})\cdots(1-x^{n+1+k}) + \cdots \end{aligned}$$

coincides, by definition, with P_{n+1}. $A_0 - B_0 + A_1 = 1 - x^n + x^n - x^{2n+1} = 1 - x^{2n+1}$. As a result we obtain the relation

$$P_n = 1 - x^{2n+1} - x^{3n+2}P_{n+1}. \tag{8}$$

Our process of expanding product (1) into a series has been completely described. It remains to see what comes out as a result. Express P_{n-1} in terms of P_n, and substitute expression (8) for P_n. We obtain

$$P_{n-1} = 1 - x^{2n-1} - x^{3n-1}(1 - x^{2n+1} - x^{3n+2}P_{n+1}).$$

In the same manner express P_{n-2} in terms of P_{n-1} and substitute the above expression for P_{n-1}. We obtain

$$P_{n-2} = 1 - x^{2n-3} - x^{3n-4}(1 - x^{2n-1} - x^{3n-1}(1 - x^{2n+1} - x^{3n+2}P_{n+1})).$$

In n steps we come to P_0 and obtain an expression containing pairs of summands $1 - x^{2n+1}$ with alternating signs, while the expression $1 - x^{2n+1}$ enters with the sign $(-1)^n$. Each expression of this kind has to be multiplied by a certain power of x. Namely, when passing from P_n to P_{n-1} there appears the factor x^{3n-1}, when passing from P_{n-1} to P_{n-2}—the factor x^{3n-4}, etc. As a result, the sum $1 - x^{2n+1}$ will enter into the expression for P_0 with the factor $x^{2+5+\cdots+(3n-1)}$. The exponent of this power is the sum of an arithmetic progression $2+5+\cdots+(3n-1) = \dfrac{3n^2+n}{2}$. Thus, product (1) is equal to the sum of terms $(-1)^n x^{\frac{3n^2+n}{2}}(1 - x^{2n+1})$. We see that $\dfrac{3n^2+n}{2}$ is the pentagonal number corresponding to the value n, and $\dfrac{3n^2+n}{2} + 2n + 1 = \dfrac{3(n+1)^2 - (n+1)}{2}$ is the pentagonal number corresponding to the value $-(n+1)$. Therefore the product (1) is equal to the sum of terms $(-1)^n x^{\frac{3n^2+n}{2}}$ for $n = 0, -1, 1, -2, 2$, etc. This is exactly the assertion of Euler's Theorem.

Note that we could skip deducing the expression for P_1 by the use of P_2 (i.e., formula (7)) and only deduce formula (8), since formula (7) is its special case for $n = 1$. We made the same argument twice, just to make the logic of our reasoning clearer.

We shall present now the second proof of the Pentagonal Theorem. It is based on an identity found in XIX century by Gauss and Jacobi. It is concerned with evaluating the infinite product

(9) $$(1+xz)(1+xz^{-1})(1+x^3z)(1+x^3z^{-1})\cdots(1+x^{2n-1}z)(1+x^{2n-1}z^{-1})\cdots,$$

where the powers of x run through all odd numbers. This is an expression of a more complicated nature than we met earlier, because there are negative, as well as positive powers of z in it. We shall first convince ourselves that this expression has a meaning in the same manner as an infinite product of power series. If we consider the first n factors in product (9), then we obtain the expression

(10) $$(1+xz)(1+xz^{-1})(1+x^3z)(1+x^3z^{-1})\cdots(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}),$$

being a usual algebraic fraction. Removing all parentheses, we obtain terms of the form $x^m z^r$, where m takes positive values, and r both positive and negative. If we consider the following factors of product (9), then after removing the parentheses there will appear only terms containing x with powers greater than $2n$. Thus, the coefficient of z^r will be a power series in x, and to evaluate its terms with degrees not exceeding $2n$, it is sufficient to consider the finite product (10). Hence, after expanding, infinite product (9) will be a sum of expressions $A_r(x)z^r$, where $A_r(x)$ are power series in x, and r takes all integer values. But, taking into account the symmetry of expression (9) with respect to z and z^{-1}, the expression obtained after expanding will also be symmetric, and so the coefficient $A_r(x)z^r$ of z^r, $r > 0$, will be equal to the coefficient $A_{-r}(x)$ of z^{-r}. As a result, the whole product (9) can be written in the form

(11) $$A_0(x) + A_1(x)(z+z^{-1}) + \cdots + A_r(x)(z^r + z^{-r}) + \cdots.$$

Our task is in fact to evaluate the power series $A_0(x)$, $A_1(x)$, … We shall do this in two steps.

The first step is completely analogous to the reasoning used in Sec. 3. Denote product (9) by $F(z)$ and replace z with x^2z, i.e., consider $F(x^2z)$. Each factor $1+x^{2k-1}z$ or $1+x^{2k-1}z^{-1}$, when z is replaced by x^2z, will give a factor $1+x^{2k+1}z$ or $1+x^{2k-3}z^{-1}$ of similar kind. Thus, factors in $F(z)$ only change their places and this product will not change, except that:

a) there will be no factor $1+xz$ (all factors $1+x^{2k+1}z$ will have exponents $2k+1 \geqslant 3$);

b) from the factor $1+xz^{-1}$ there will appear a factor $1+x^{-1}z^{-1}$ which wasn't there before.

All this can be written as one formula

$$F(x^2z)\frac{1+xz}{1+x^{-1}z^{-1}} = F(z).$$

But, obviously, $\dfrac{1+xz}{1+x^{-1}z^{-1}} = xz$ and we can write this formula as

$$F(x^2z)xz = F(z). \tag{12}$$

Recall now that we can consider the product $F(z)$ in the form (11) and apply relation (12) to this representation. We obtain

$$\begin{aligned}(A_0(x) + A_1(x)(x^2z + x^{-2}z^{-1}) + \cdots + A_r(x)(x^{2r}z^r + x^{-2r}z^{-r}) + \cdots)xz =\\ = A_0(x) + A_1(x)(z+z^{-1}) + \cdots + A_r(x)(z^r+z^{-r}) + \cdots.\end{aligned}$$

Equate the terms containing z^r. On the left-hand side they are obtained from the term containing z^{r-1}, after multiplying by xz. That is, from the term $A_{r-1}(x)x^{2(r-1)}z^{r-1}$ after multiplying by xz. On the right-hand side—from the term containing z^r, i.e., from $A_r(x)z^r$. As a result we obtain that

$$A_{r-1}(x)x^{2r-1} = A_r(x).$$

We see that all series $A_r(x)$ are expressed in terms of each other. In particular,

$$A_r(x) = x^{2r-1}A_{r-1}(x) = x^{2r-1+2r-3}A_{r-2}(x) = \cdots = x^{2r-1+2r-3+\cdots+1}A_0(x). \tag{13}$$

In the exponent of x, there is the sum of the first r odd numbers, $1+3+\cdots+(2r-1) = r^2$. So, we can rewrite relation (13) as

$$A_r(x) = x^{r^2}A_0(x).$$

It is possible to prove that considering the terms with negative powers of z gives the relation of the same kind, but we shall not do that here.

We see that the factor $A_0(x)$ can be taken out of the whole expression (11) and we obtain for our product (9) a very elegant expression

$$\begin{aligned}(1+xz)(1+xz^{-1})(1+x^3z)(1+x^3z^{-1})\cdots(1+x^{2r-1}z)(1+x^{2r-1}z^{-1})\cdots =\\ = A_0(x)(1+x(z+z^{-1})+\cdots+x^{r^2}(z^r+z^{-r})+\cdots),\end{aligned} \tag{14}$$

but the factor $A_0(x)$ in it still remains undetermined.

The given argument follows completely the method applied in Sec. 3 for evaluating product (33). But there we had a (constant) term known in advance and all other terms could be expressed in terms of it. In the present case there is no such term and so there appears the factor $A_0(x)$ in formula (14) which still has to be determined.

So we pass now to the second step of our proof—evaluating the series $A_0(x)$. Recall that, in order to find the terms of that series not exceeding $2n$, it is enough to consider finite product (10). Some of the coefficients in it can be found explicitly. For example, the coefficient of z^n is obtained if the summand $x^{2r-1}z$ is taken from all parentheses of the form $1+x^{2r-1}z$, and the summand 1 is taken from parentheses $1+x^{2r-1}z^{-1}$. As a result we obtain the term $x^{1+3+\cdots+(2n-1)}z^n = x^{n^2}z^n$. Applying the same method as in the first step of the proof, we can express all terms using this one, and so $A_0(x)$ can be expressed in the same terms. This is in fact our plan of proof.

Denote product (10) (for some fixed n) by $f(z)$ and replace again z by x^2z. We obtain more changes this time than with the same transformation of product (9). Namely, there appear, as before, changes in the beginning of the product: there will be no factor $1 + xz$ in $f(x^2z)$ and there is a new factor $1 + x^{-1}z^{-1}$ which did not appear in $f(z)$. Moreover, there will be changes at the end of product (11): a new factor $1 + x^{2n-1} \cdot x^2z = 1 + x^{2n+1}z$ appears, and the factor $1 + x^{2n-1}z^{-1}$ vanishes (after the substitution of x^2z instead of z, the exponent of x gets smaller). The rest of the factors in $f(x^2z)$ and $f(z)$ will be the same. We again obtain a relation between them, only this time a little bit more complicated:

$$f(x^2z)\frac{1+xz}{1+x^{-1}z^{-1}}\frac{1+x^{2n-1}z^{-1}}{1+x^{2n+1}z} = f(z). \tag{15}$$

As we have seen, $\dfrac{1+xz}{1+x^{-1}z^{-1}} = xz$, $xz(1+x^{2n-1}z^{-1}) = xz+x^{2n}$ and relation (15) acquires the form

$$f(x^2z)(xz+x^{2n}) = f(z)(1+x^{2n+1}z). \tag{16}$$

Represent now $f(z)$ in the form of a power series in z and z^{-1}:

$$f(z) = a_0(x) + a_1(x)(z+z^{-1}) + \cdots + a_n(x)(z^n+z^{-n}), \tag{17}$$

and substitute this expression into relation (16):

$$(a_0(x) + a_1(x)(x^2z + x^{-2}z^{-1}) + \cdots + a_n(x)(x^{2n}z^n + x^{-2n}z^{-n}))(xz + x^{2n}) =$$
$$= (a_0(x) + a_1(x)(z+z^{-1}) + \cdots + a_n(x)(z^n+z^{-n}))(1+x^{2n+1}z).$$

Equate coefficients of z^r on both sides of the equality (taking $r \geqslant 1$). On the left-hand side, such term is obtained from the term containing z^{r-1}, after multiplying by xz and from the term containing z^r after multiplying by x^{2n}. On the right-hand side such term is obtained from the term containing z^{r-1} after multiplying by $x^{2n+1}z$, and from the term containing z^r after multiplying by 1. As a result we obtain the relation

$$a_{r-1}(x)x^{2r-1} + a_r(x)x^{2r+2n} = a_{r-1}(x)x^{2n+1} + a_r(x).$$

Written in another way,

$$a_{r-1}(x)x^{2r-1}(1-x^{2n-2r+2}) = a_r(x)(1-x^{2n+2r}). \tag{18}$$

Relation (18) gives enables us to express coefficients $a_r(x)$ in terms of each other. For example,

$$a_r(x) = \frac{a_{r-1}(x)x^{2r-1}(1-x^{2n-2r+2})}{1-x^{2n+2r}}.$$

Replacing in (18) r by $r-1$, we express in the same manner $a_{r-1}(x)$, and substituting it we obtain

$$a_r(x) = \frac{a_{r-2}(x)x^{2r-1+2r-3}(1-x^{2n-2r+2})(1-x^{2n-2r+4})}{(1-x^{2n+2r})(1-x^{2n+2r-2})}.$$

Repeating the process r times, we obtain in the numerator the power of x equal to $x^{(2r-1)+(2r-3)+\cdots+1} = x^{r^2}$. Therefore,

$$a_r(x) = a_0(x)x^{r^2}\frac{(1-x^{2n-2r+2})(1-x^{2n-2r+4})\cdots(1-x^{2n})}{(1-x^{2n+2r})(1-x^{2n+2r-2})\cdots(1-x^{2n+2})}.$$

Since we know the coefficient $a_n(x)$ (it is equal to x^{n^2}), we put in the last relation $r = n$:

$$x^{n^2} = a_0(x)x^{n^2}\frac{(1-x^2)(1-x^4)\cdots(1-x^{2n})}{(1-x^{2n+2})\cdots(1-x^{4n})}.$$

This can be written in the form

$$a_0(x) = \frac{(1-x^{2n+2})\cdots(1-x^{4n})}{(1-x^2)(1-x^4)\cdots(1-x^{2n})}. \tag{19}$$

Recall now how we have started deducing this result: in order to find the coefficients by powers of x not exceeding $2n$ in product (9), it is sufficient to find these terms in the finite product (10). In particular, it relates to the terms entering $A_0(x)$: for exponents not exceeding $2n$ they coincide with the respective terms in $a_0(x)$. But, in the numerator of formula (19) all powers of x are greater than $2n$. Hence, when evaluating the terms with degrees not exceeding $2n$, they can be eliminated, and we see that in the series $A_0(x)$ the terms with powers not exceeding $2n$ are the same as in the series

$$\frac{1}{(1-x^2)(1-x^4)\cdots(1-x^{2n})}.$$

Our conclusion is valid for each n. This proves that

$$A_0(x) = \frac{1}{(1-x^2)(1-x^4)\cdots(1-x^{2n})\cdots},$$

where in the denominator binomials $1-x^{2n}$ with all natural n are multiplied. Taken together with formula (14) it determines product (9) completely. Multiplying by the denominator and rearranging (the legitimacy of this operation was discussed in Sections 2 and 3), we obtain the relation

$$\begin{aligned}(1+xz)(1+xz^{-1})(1-x^2)\cdots(1+x^{2n-1}z)(1+x^{2n-1}z^{-1})(1-x^{2n}) = \\ = 1 + x(z+z^{-1}) + x^4(z^2+z^{-2}) + \cdots + x^{n^2}(z^n+z^{-n}) + \cdots,\end{aligned} \tag{20}$$

which is in itself very elegant.

The Pentagonal Theorem is a consequence of identity (20). Really, put $x = y^3$, and $z = -y$. Then, on the left, $1 - x^{2n} = 1 - y^{6n}$, $1 + x^{2n-1}z = 1 - y^{6n-2}$, $1 - x^{2n-1}z^{-1} = 1 - y^{6n-4}$, i.e., on the left-hand side of identity (20) products $1 - y^n$ with all even natural n appear. On the right, the term $x^{n^2}z^n$ gives $(-1)^n y^{3n^2+n}$, and the term $x^{n^2}z^{-n}$ gives $(-1)^n y^{3n^2-n}$. We obtain, both on the left and on the right, series containing just even powers of y. Thus, we can put $y^2 = t$. As a result, on the left we obtain the product of all factors $1 - t^n$ with natural n, and on the right the sum of terms $(-1)^n t^{\frac{3n^2 \pm n}{2}}$. This equation is the Pentagonal Theorem.

Problems

1. Find the representation of the product

$$(1-x)^2(1-x^2)(1-x^3)^2(1-x^4)\cdots(1-x^{2n+1})^2(1-x^{2n+2})\cdots$$

as a series.

2. Do the same for the product

$$(1+x)^2(1-x^2)(1+x^3)^2(1-x^4)\cdots(1+x^{2n+1})^2(1-x^{2n+2})\cdots.$$

3. Prove the identity

$$(1-x^2)(1+x)(1-x^4)\cdots(1+x^n)(1-x^{2n+2})\cdots = 1+x+x^3+x^6+\cdots+x^{\frac{n(n+1)}{2}}+\cdots.$$

4. Prove the identity

$$\frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = 1+x+x^6+\cdots+x^{\frac{n(n+1)}{2}}+\cdots.$$

APPENDIX II

The generating function for Bernoulli numbers

Consider a remarkable power series

$$e(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots. \tag{1}$$

It is possible to prove that numbers can be substituted for x, an hence define an important function: it is possible to prove that $e(x) = e^x$, where e is the base of the natural logarithm. We shall stay with the purely algebraic theory of power series. Nevertheless, we shall show that the series $e(x)$ possesses some properties of the exponential function. Introduce a new variable y and consider the series $e(y)$ and $e(x+y)$. We shall prove the identity

$$e(x+y) = e(x)e(y). \tag{2}$$

Really, replace x by $x+y$ in formula (1). The term of degree n will have the form $\frac{1}{n!}(x+y)^n$. Expand $(x+y)^n$ using the binomial formula and use the expressions for binomial coefficients that we found in Sec. 3, Ch. II (formula (24)):

$$\frac{1}{n!}(x+y)^n = \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!}\frac{y}{1!} + \frac{x^{n-2}}{(n-2)!}\frac{y^2}{2!} + \cdots + \frac{y^n}{n!}.$$

This is the sum of expressions of the form $\dfrac{x^k}{k!}\dfrac{y^{n-k}}{(n-k)!}$ for $k = n,\ n-1,\ \ldots,\ 0$, i.e., the sum of products of the term of degree k in the series $e(x)$ and the term of degree $n-k$ in the series $e(y)$. But, this is exactly the term of degree n in the series $e(x)e(y)$. This proves formula (2).

Essentially, formula (2) is equivalent to the binomial formula and it contains in itself *all* binomial formulas for all values of n.

Using this important property of the series $e(x)$ we can construct new types of generating functions. Let a be a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots)$. Introduce the series

$$e(ax) = \alpha_0 + \frac{\alpha_1 x}{1!} + \frac{\alpha_2 x^2}{2!} + \cdots + \frac{\alpha_n x^n}{n!} + \cdots .$$

It is called the *factorial generating function* of the sequence a. If addition of sequences is defined term by term, i.e., if for $a = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots)$ and $b = (\beta_0, \beta_1, \ldots, \beta_, \ldots)$, $a+b = (\alpha_0+\beta_0, \alpha_1+\beta_1, \ldots, \alpha_n+\beta_n, \ldots)$, then,

(3) $$e((a+b)x) = e(ax) + e(bx).$$

We extend to power series notation introduced in the Appendix to Chapter II. Namely, if $f(t,x) = f_0(t) + f_1(t)x + \cdots + f_n(t)x^n + \cdots$ is a power series whose coefficients are polynomials, and a is a sequence, put

$$f(a,x) = f_0(a) + f_1(a)x + \cdots + f_n(a)x^n + \cdots .$$

The meaning of the expression $f(a)$ when $f(t)$ is a polynomial was defined in the Appendix of Ch. II: if $f(t) = a_0 + a_1 t + \cdots + a_m t^m$ and $a = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots)$, then $f(a) = a_0 + a_1\alpha_1 + \cdots + a_m\alpha_m$. In this notation, the factorial generating function of a sequence a is written as $e(ax)$. It is easy to see that an analogue of relation (2) holds:

(4) $$e((\alpha + a)x) = e(\alpha x)\cdot e(ax).$$

Here α is a number, $a = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots)$ a sequence, and $\alpha + a$ denotes the sequence $(\alpha_0+\alpha, \alpha_1+\alpha, \ldots, \alpha_n+\alpha, \ldots)$. The proof is the same as for relation (2). The term of degree n on the left-hand side is by definition equal to $\dfrac{1}{m!}(\alpha+a)^m$, i.e., to the sum of terms $\dfrac{1}{m!}\dfrac{m!}{k!\,(m-k)!}\alpha^k\alpha_{n-k}x^n = \dfrac{1}{k!}\alpha^k x^k\dfrac{1}{(n-k)!}\alpha_{n-k}x^{n-k}$. But this is the product of the term of degree k in $e(\alpha x)$ with the term of degree $n-k$ in $e(ax)$. By the definition of multiplication of power series, the sum of all these terms is the term of degree n in the product $e(\alpha x)e(ax)$. This proves relation (4).

Let us find, using the above formulae, the factorial generating function for the sequence of Bernoulli numbers $B = (B_0, B_1, \ldots, B_n, \ldots)$. Recall that Bernoulli numbers are defined by the relation

$$(B+1)^m - B_m = m, \qquad m = 1, 2, \ldots \tag{5}$$

Consider the factorial generating functions of the three sequences entering relation (5). These are the sequence $(1+B)^m$, the sequence B^m and the sequence $(0, 1, \ldots, n, \ldots)$ (formed by the right-hand sides). Denote the last sequence by N. Using property (3), we can write down all relations (5) in the form

$$e((1+B)x) - e(Bx) = e(Nx), \tag{6}$$

and using property (4), $e((1+B)x) = e(x)e(Bx)$. It remains to find the series $e(Nx)$. Its term of degree n is equal to $\frac{n}{n!}x^n = \frac{1}{(n-1)!}x^{n-1} \cdot x$. Therefore, $e(Nx) = xe(x)$ and relation (6) acquires the form $e(Bx)(e(x) - 1) = xe(x)$, wherefrom

$$e(Bx) = \frac{xe(x)}{e(x) - 1}. \tag{7}$$

This is the form of the factorial generating function for Bernoulli numbers. Note that in the denominator there is the series $e(x) - 1$, with the constant term equal to 0. The factor x can be taken out of this series, and then cancelled with the same factor in numerator, and the remaining power series has the constant term equal to 1 and so it has the inverse by Theorem 1 (Sec. 2).

All the properties of Bernoulli numbers can be easily deduced from this form of the generating function. Let us prove, for example, that all Bernoulli numbers with odd indices are equal to 0, except for B_1 (Problem 3 in Appendix to Ch. II). As we know, $B_1 = 1/2$ (this follows easily from formula (6)). Thus, our assertion means that the series $e(Bx) - \frac{x}{2}$ contains just terms with even powers of x. If we replace x in a power series $f(x)$ by $-x$, then the terms with even powers of x do not change, and the terms with odd powers change sign. The fact that there are only terms with even degrees in the power series $f(x)$ is equivalent to the fact that $f(-x) = f(x)$.

Thus, we have to convince ourselves that the series $e(Bx) - \frac{x}{2}$ does not change when x is replaced by $-x$. Using expression (7) for the series $e(Bx)$, we obtain that our assertion is equivalent to the identity

$$\frac{xe(x)}{e(x) - 1} - \frac{x}{2} = \frac{-xe(-x)}{e(-x) - 1} + \frac{x}{2}.$$

This equality can be cancelled by x and $\frac{1}{2}$ can be transferred to the left-hand side. Denote $e(x)$ by u. According to identity (2), $e(-x) = u^{-1}$. Our equality acquires the form $\dfrac{u}{u-1} - 1 = \dfrac{u^{-1}}{u^{-1} - 1}$, which is evident.

Let us now demonstrate a connection between Bernoulli numbers and the sums of powers of consecutive natural numbers $S_m(n) = 1^m + 2^m + \cdots + n^m$. In the Appendix to Ch. II the formula $S_m(n) = \frac{1}{m+1}((B+n)^{m+1} - B_{m+1})$ was obtained, which, replacing m by $m - 1$, can be written as

$$S_{m-1}(n) = \frac{1}{m}((B+n)^m - B_m). \tag{8}$$

Let us show this in a different way.

Consider the factorial generating function of the sequence $(B+n)^m - B_m$ (for fixed n). Using property (3) it can be written in the form $e((B+n)x) - e(Bx)$. According to identity (4), this series is equal to $e(Bx)e(nx) - e(Bx) = e(Bx)(e(nx)-1)$. It follows from property (3) (by induction on n) that $e(nx) = e(x)^n$. Substituting this value into expression (7) for $e(Bx)$, we can write our series in the form $xe(x)\dfrac{e(x)^n-1}{e(x)-1}$. According to identity (12) of Ch. I we have

$$\frac{e(x)^n-1}{e(x)-1} = 1 + e(x) + \cdots + e(x)^{n-1}$$

and therefore

$$xe(x)\frac{e(x)^n-1}{e(x)-1} = x(e(x) + \cdots + e(x)^n).$$

Replacing again $e(x)^k$ by $e(kx)$ for all $k = 1, \ldots, n$ (by property (3)), we obtain

$$xe(x)\frac{e(x)^n-1}{e(x)-1} = x(e(x) + e(2x) + \cdots + e(nx)).$$

Let us find the coefficient of x^m in the series on the right. It is equal to the coefficient of x^{m-1} in the series $e(x) + e(2x) + \cdots + e(nx)$. In $e(kx)$, the coefficient of x^{m-1} is equal to $\dfrac{k^{m-1}}{(m-1)!}$, and in the whole sum, to the expression

$$\frac{1}{(m-1)!} + \frac{2^{m-1}}{(m-1)!} + \cdots + \frac{n^{m-1}}{(m-1)!} = \frac{S_{m-1}(n)}{(m-1)!}.$$

Put $\alpha_m = (B+n)^m - B_m$ and denote the sequence $(\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots)$ by a. We have proved that the coefficient of x^m in the series $e(ax)$ is equal to $\dfrac{S_{m-1}(n)}{(m-1)!}$. By the definition, it is equal to $\dfrac{\alpha_m}{m!}$. Therefore

$$\frac{\alpha_m}{m!} = \frac{S_{m-1}(n)}{(m-1)!},$$

wherefrom identity (8) follows immediately.

Problems

1. Define the sequence B'_n, where $B'_1 = -\frac{1}{2}$, $B'_n = B_n$ for $n \geqslant 2$. Prove that the factorial generating function of the sequence B'_n has the form $e(B't) = \dfrac{t}{e^t-1}$. Prove for the sequence B'_n the relation $(B'+1)^m = B'_m$ for $m \geqslant 2$.

2. Check the relation $e((B-\frac{1}{2})x) = 2e(B\frac{x}{2}) - e(Bx)$.

3. Prove that for m even, Bernoulli polynomial $B_m(x)$ has a root $x = -\frac{1}{2}$. [*Hint*. Use Problems 2 and 3 in Appendix of Ch. II.]

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