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LOOKING AT FUNCTION CONCEPT THROUGH INTERACTIVE ANIMATIONS

Martti E. Pesonen, Lenni Haapasalo and Hanna Lehtola

Abstract. If we agree that the main goal of mathematics education is to develop both procedural and conceptual knowledge and to make links between the two, an important research domain is the analysis of appropriate pedagogical solutions achieving this goal at the university level. This paper presents CAL activities for enhancing university freshmen's understanding of the function concept, and summarizes some outcomes of these activities concerning these knowledge types.

AMS Subject Classification: 00 A 35.

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1. Searching for a bridge between school and university

Tertiary mathematics education seems to have a big concern in most institutions. In departments of mathematics professors struggle with freshmen's poor conceptual understanding, whereas other departments or institutions, trying to teach more or less context-based applied mathematics, face problems by trying to help students achieve an appropriate level of procedural knowledge. These quite global problems have been recognized among the international scientific community, as well. There are hundreds of papers concerning this hot topic, and several symposiums have been organized, e.g. by ICMI. One of the basic questions in Singapore conference in 1998 (Holton, 2001) was the gap between school and university. The state of mathematics education can perhaps still be described like Griffiths & Oldknow (1993, p. 1): " ... mathematics has now become a rather strange study to embark upon. For many within traditional universities, it consists of a number of discrete courses with titles such as 'numerical analysis', 'linear algebra', 'probability theory', and so on, which are frequently studied with little interdependence. It is not unusual for a student to complete a course of three years undergraduate study of the subject called 'mathematics' and yet to have little idea of what 'the subject' really is except the name of a set of parts ... "

Possible reasons for this situation can be found in the basic arguments of the Joint European Project MODEM (`http://www.joensuu.fi/lenni/modem.html`): "University mathematicians have tended to take the view that, at any given level, there is a set of basic skills and concepts which must be learned and practiced before any engagement can be attempted with the actual practice of mathematics. Within that university tradition there is also a widespread single style of teaching and learning. Mathematics tends to be explained as an organized body of knowledge, in which students are largely passive, practicing old, clearly formulated, and unambiguous questions for timed examinations. The large body of theory is found to be abstract and depends on an unfamiliar language. These features are of course essential for the purposes of a professional mathematician, but they leave many students dispirited and bored, and their performance in more advanced courses is poor because the foundations are weak: the examiners are reduced to setting only bookwork or stereotyped questions, which can be remembered without becoming a vital part of the student"

The leading question, coming up implicitly from the above criticism, is the dilemma between conceptual (C) and procedural (P) knowledge: Does the student have to understand for being able to do, and vice versa (Haapasalo, in preparation). A comprehensive theoretical framework for the approaching to this basic question can be found in Haapasalo & Kadijevich (2000), or Kadijevich & Haapasalo (2001) as far as CAL is concerned.

For representing the case study in this paper, let's take a look on studies concerning one of the basic topics in the university freshmen's mathematics: the concept of function[1]. Tall & Bakar (1991) make conclude that "the learner cannot construct the abstract concept of function without experiencing examples of the function concept in action, and the students cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept". Support for this statement can be found in many other studies as Breidenbach et al. (1992), Tall (1992), Vinner & Dreyfus (1989), and Brown et al. (1997). Using a recent analysis of Haapasalo & Kadijevich (2000) this can be expressed even more generally as follows.

The dominance of P over C seems quite natural both in the development of scientific and individual knowledge. So, an appropriate pedagogical idea in any topic could be to go for spontaneous procedural knowledge. The logical relation between P and C in this so-called developmental approach is based upon genetic view (i.e. P is necessary for C) or simultaneous activation view (i.e. P is necessary and sufficient for C). On the other hand, it seems appropriate to claim that the goal of any education should be to invest on conceptual knowledge from the first beginning. If so, the logical basis of this so-called educational approach is dynamic interaction view (i.e. C is necessary for P), or again the mentioned simultaneous activation view. The latter means that the learner has opportunities to activate conceptual and procedural features of the current topic simultaneously. By "activating" we mean certain mental or concrete manipulations of the representatives of each type of knowledge. Being in the intersection of two complementary approaches, the simultaneous activation view is loaded with some expectations concerning the planning of learning environments. Modern technology, of course, offers natural solutions for these kinds of activities.

[1]Note that there is a question about a typical precept in the sense of Gray & Tall (1993): a combined mental object consisting of a process, a concept produced by that process, or a symbol which may be used to denote either of both.

If we accept that a viable understanding of the function concept constitutes a necessary background for higher mathematics, we have to try to allow students to come up with their procedural ideas and finally help them to understand the conceptual features of the function, as well. This means searching a certain balance between developmental approach and the educational one, building a bridge between more or less procedural school thinking and conceptual academic thinking. For the teacher education, there is one more big challenge: how to reach both a good mathematical understanding and an appropriate pedagogical know-how in the sense of modern theories of teaching and learning? At our University, the Department of Mathematics takes care of the mathematical education of secondary and tertiary level mathematics and science pre-service teachers. General education and pedagogy of mathematics are traditionally taught in the Faculty of Education. However, there is a strong tendency to integrate pedagogical issues in the subject studies. Furthermore, in Finland there are increasing demands to use technology at all levels of education.

2. Case study with interactive Javasketchpad problems

Before freshmen in our Department of Mathematics learned anything about functions after their secondary school, they had a three-hour working period with Javasketchpad applets[2]. By a sketch we mean dynamical interactive applet constructions containing text parts, figures, and geometrical elements (points, lines, rays, segments, circles and more advanced constructions) to be manipulated with mouse. Control buttons could be used for showing, hiding, moving and animating (Fig. 1).

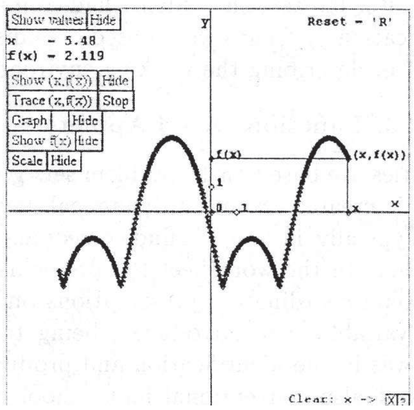

Fig. 1. Screen shot of an interactive applet in the Function Demo

Sketch codes could be generated automatically by Geometer's Sketchpad, but for a more precise output and control of the command structures they were pro-

[2]The computer activities could also be done through distance learning.

grammed by hand using the Java- sketchpad construction language. The sketches together with the appropriate problem sets were shown by means of an html-browser. The answers were sent directly to the teacher by using a www form. The generated *Form Mail* document showed to the student the correct answers and their own answers. Altogether 74 email answers to the problems were received from students participated in this case study.

The applets were from the first beginning aimed to whitewash students' naïve and stereotypic conceptions based on school mathematics, and to increase their sensitivity to use different glasses when looking at mathematical objects. The applets did not offer any thorough-going learning environment for the function concept, of course. However, for developing such kind of CAL environment later on, the applet construction has to be based on appropriate pedagogical ideas. We applied the theories of concept images and concept definitions (e.g. Vinner & Dreyfus 1989; Vinner 1991) combined with the framework of MODEM educational approach[3] (Haapasalo 1993, 1997). The latter means an investment on conceptual knowledge utilizing a systematic framework involving five phases of concept building (orientation, definition, identification, production, and reinforcement), and pointing out links between different types of representation forms (verbal, symbolic, graphic). In our "evolution version" of applets, mainly task types of concept identification were in focus, utilizing some basic features of simultaneous activation principle described above.

The working period was started by a paper-and-pencil pre-test consisting mainly of simple identification tasks. The students had to identify weather the function properties were fulfilled when a candidate was in a verbal or graphic form. Right after the pre-test there was a two-hour computer activity called *Function Demo*[4]. A week after such a computer lab session students had to make an online post-test where again some identification problems were posed. We discuss and compare the results of the two tests after describing the working environment with the applets.

3. Function Demo Applets

The computer activities are based on 17 problem sets given with a www worksheet. Each problem set is created around a dynamical Javasketchpad figure and contains 3–8 questions. Typically, in order to find correct answers, some interaction with the applet was needed. In the worksheet the problems range from set-to-set arrow pictures to traditional coordinate representations on a plane. Also vector-valued functions of one variable were introduced, being totally new for most of the students. The focus was in the identification and production tasks concerning the function definition, but also conventional high school mathematics problems were included. The seventeen problem sets (S) were integrated in the worksheet structure as follows:

[3]The theory is demonstrated effortlessly by another CAL environment, the software being freely downloadable at `http://www.joensuu.fi/lenni/programs.html`

[4]Hyperlinks to the tests and an English version of the Function Demo (equipped with student scores) are found at the URL `http://www.joensuu.fi/mathematics/MathDistEdu/Paper2003/FunctionConcept.htm`

I *Definition of function: domain—rule—range (S1–S5)*

- *Definition and terminology (text to be read and studied by the student)*
- *Different representations, bijection*

These first sketches are meant to familiarize the student with the dynamical nature of the pictures and to get oriented in the function concept. In Sketch 5 the idea of a dynamical (discrete) variable appears, building a bridge to the idea of continuity. The questions deal mostly with identifying functions and producing verbal or symbolical counterparts for the graphical representations.

II *About graphical representations of real functions (S6–S15)*

- *One dimensional axis-to-axis representation, Resume*

The functions act between two parallel real lines, the upper containing the domain and the lower the function values, respectively. We call this axis-to-axis representation. All domains are intervals, but the ranges may be intervals, discrete point sets, multiple points or exceed the real line. For arranging the restriction to an interval domain two ways to are used: either the variable is bound to an interval or there is no image when the variable goes outside the domain. It is not yet clear which way is better. In different sketches various tools are available:

- S6: Function: continuous, periodic; Buttons: animate, ticks, restrict, trace
- S7: Function: discontinuous, discrete; Buttons: values, ticks, trace
- S8: Function: continuous; Buttons: values, ticks, trace
- S9: Relation: two-valued; Buttons: values, ticks
- S10: Function, not real-valued; Buttons: animate, ticks

Sketches in S6–S8 are concerned with properties that students should know from school: domain, co-domain, image/range, values, extrema, monotonicity, and continuity. Questions in S9–S10 are again about identifying functions. Section II is finished with a query of student opinions on how the axis-to-axis representation suits to several kind of familiar problems about functions.

- *Plane curve representation*

Behind the sketches in S11 and S12 there is exactly the same function, and precisely the same questions are posed. S11 tries to serve as a link between the axis-to-axis representation and the conventional coordinate system illustration of real functions. The latter provides the student with tracing the (variable, image) point and the familiar function graph. Several kinds of control buttons and sliding value points are present here, as in the sketches to come.

- *Examples (power, root, exp, log)*

S13–S15 ar1e probably closest to school mathematics.

III *About vector valued functions of a real variable (S16–17)*

- *Plane curve and motion in the plane*

The last two sketches try to open and expand the students' view on the role the Euclidean plane when representing functions. The problems are mostly of experimental or measuring nature, but the basic idea of variable value and image value are pointed out.

IV *Student feedback* (text area for free expressions)

4. Discussion based on observations and students' test scores

The tests and the Function Demo were planned together by Pesonen and Lehtola. Some of the results are based on a Master's Thesis (Lehtola, 2002) or the material gathered for it. We would like to point out that the main purpose of this case study was to get a priori information for developing our worksheet into a sophisticated CAL environment for the function concept and for carrying out later a more systematic study, including qualitative analysis, as well. Hence, the test tasks were not planned to cover the measurement of the conceptual understanding, as should be the case if applying the *MODEM* instruments[5] (Haapasalo 1993, 1997). For making the comparison of the two tests possible in our case study, we had to look for the same task types in both tests. Unfortunately only simple identification tasks, measuring mainly just understanding of the function definition, happened to remain for this purpose.

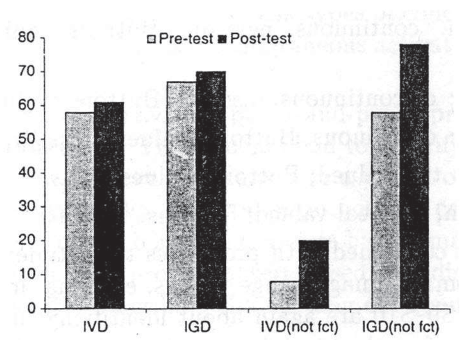

Fig. 2. Improvements in students' scores (percentage of the correct answers)

In Fig. 2 these task types are denoted by IVD and IGD. The former means that students have to identify if the function definition was satisfied when the candidate is in a verbal form, and is, in the latter case, given graphically. In case that the definition is satisfied, students seem to reach a satisfying level of identification. Their performances did not improve significantly, when concerning the average scores within each task type. More interesting are the two pairs of bars on the right, showing a case when the candidate is not a function at all. Working with our applets improved students' performances significantly within both forms of representation. One could ask why the bars IVD are so low when comparing with those of IGD, now. A possible explanation comes from MODEM studies. Both

[5]All empirical studies within MODEM paradigm have revealed that concept understanding is most reliably measured through production tasks, and less unreliably through identification tasks.

on the level of concept identification and concept production, verbal tasks were major predictor of conceptual understanding (Haapasalo 1993, 1997). If students would have conceptual knowledge on functions, they should be able to identify anti-examples in verbal form, as well. The message of Fig. 2 might be that function is probably handled in school mathematics more or less procedurally without trying to conceptualize it. Indeed, when in the pre-test students were asked to write down the function definition, the distribution of their expressions was: correct 25%, half correct 25%, invalid 40%, no answer 10%. A clear message of the outcome is that one should be humble when meeting challenges of a successful CAL environment with much richer tasks types than what was the case with our applets.

The worksheet ended with an open ended *Student Feedback* text area starting: "I think that … ". Fig. 3 demonstrates students' ($N = 74$) expressions and their frequencies showing that the feedback was mostly positive, although some criticism was given, as well.

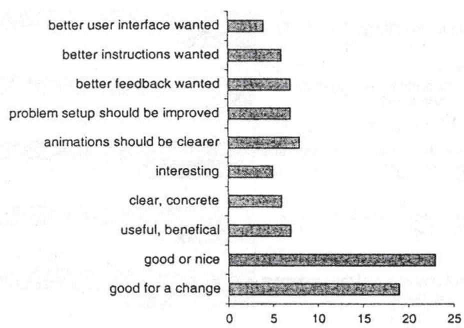

Fig. 3. Students' free expressions and their frequencies

In the post-test a structured questionnaire was made about one example problem set (see Fig. 4) and the Function Demo worksheet in general. Fig. 5 shows the percentage of each chosen opinion. Statement-wise Binomial Tests show that there were significantly more positive than negative opinions. A deviation can be found concerning the answers to the second and to the last question (Chi Square Test). The former is significantly more positive and the latter significantly less positive than the opinions in general.

As regards our experience when conducting this case study, many new dynamical ideas for developing of mathematics instructions on the university level were rising, not only concerning this restricted topic but also in more general. Students should be given opportunities to get free from their, often very stereotypic and naive, mathematical conceptions. To achieve this end, teachers should utilize suitable software- hardware combinations that allow constructions of a mental bridge between concrete and abstract objects through simple activities, connecting symbolic (algebraic) and graphic (geometric) representations. From the learner's point of view, these representations can very often be interpreted as conceptual vs. procedural, respectively. Studying mathematics at university level is mainly based

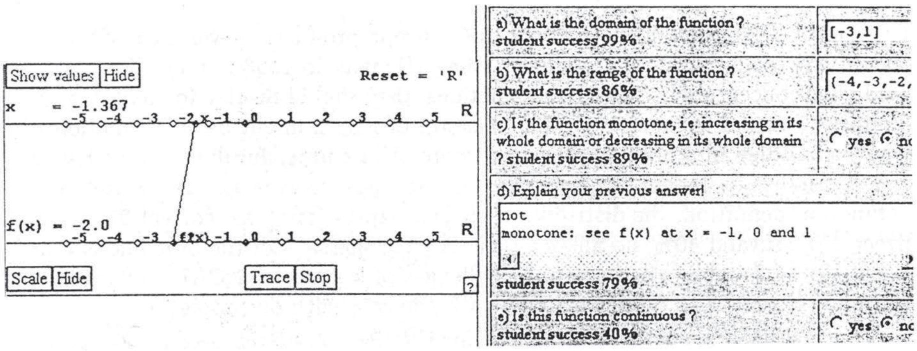

Fig. 4. Screen shot from post-test

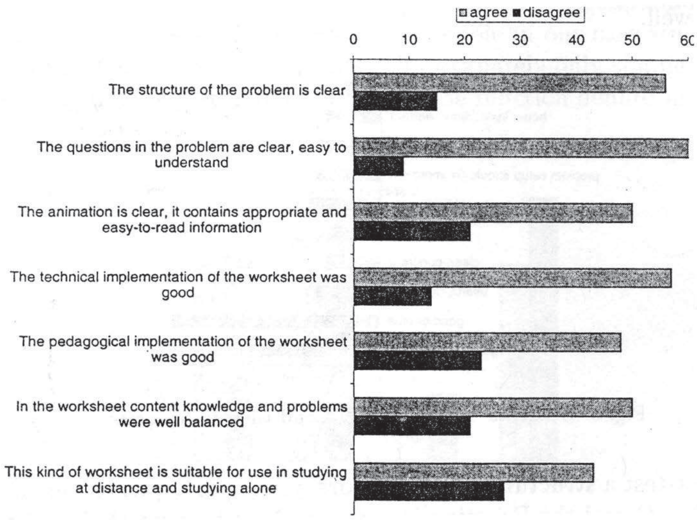

Fig. 5. The frequencies of students' answers to the structured questionnaire

on educational approach causing us big challenges to organize the relation between conceptual and procedural knowledge. This pedagogical task should be taken as seriously if our students are going to be mathematicians, engineers or teachers. In the latter case, for example, modern ideas on teaching and learning could become a viable practical theory of teachers, if same kind of processes, including new technological solutions would be a vital part of their own studies.

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Lenni Haapasalo, University of Joensuu, Department of Applied Education, P.O. Box 11, 80101 Joensuu, Finland; e-mail: lenni.haapasalo@joensuu.fi; http://www.joensuu.fi/lenni

Hanna Lehtola, University of Joensuu, Department of Mathematics, P.O. Box 11, 80101 Joensuu, Finland; e-mail: hlehtola@cc.joensuu.fi

Martti E. Pesonen, University of Joensuu, Department of Mathematics, P.O. Box 11, 80101 Joensuu, Finland; e-mail: martti.pesonen@joensuu.fi; http://www.joensuu.fi/mathematics/personnel/PesonenM.html