THE TEACHING OF MATHEMATICS
2002, Vol. V, 2, pp. 91–98

ARE QUANTITATIVE AND QUALITATIVE REASONING RELATED?

A ninth-grade pilot study on multiple proportion

Đorđe Kadijević

Abstract. The major objective of this study was to determine whether quantitative and qualitative reasoning are related, and, if so, which kind of instruction promotes their relation. The study had a pre-test/post-test design with two parallel groups. Both groups solved quantitative and qualitative problems, but while the QN group was only taught how to solve quantitative problems, the QL group was exclusively taught how to solve qualitative problems. The study used a sample of 68 ninth-grade Gymnasium (high-school) students of average mathematical abilities. The students solved multiple proportion problems. The study showed that: (a) the examined problems were hard for most students; (b) even in the QN group, quantitative reasoning was not improved; (c) qualitative reasoning was improved in both treatment groups and the QL group scored better; and (d) only the QL treatment related quantitative and qualitative reasoning.

AMS Subject Classification: 00 A 35.

Introduction

It is generally accepted that qualitative reasoning denotes reasoning about problem components and their relations by using qualitative not quantitative terms. As an example, consider the following problem:

> David did more laps than George. If David was running in a shorter time, who ran faster?

Although qualitative reasoning significantly influences problem solving performance, it has rarely been an object of scientific inquiry, especially regarding its relation to traditionally fostered quantitative reasoning. A recent survey of this important, yet neglected research area can, for example, be found in Behr et al. [1].

As for teaching for qualitative reasoning, it is generally believed that qualitative reasoning should precede quantitative exercises, as the former can guide and serve as a check for the latter. According to Behr et al. [1], "qualitative reasoning is helpful (not completely necessary) but certainly not sufficient for successful performance on quantitative proportional-reasoning problems." (p. 320) However, the evidence to date is slight. The major objective of this study was to determine whether quantitative and qualitative reasoning are related, and, if so, which kind of instruction promotes their relation.

The study dealt with multiple proportion problems such as:

> Five workers planted 120 seedlings in 8 hours. How many workers will be needed if 210 seedlings are to be planted in 7 hours?

These problems have not been extensively studied up to date despite the fact that they still offer a powerful structure for conceptualising many real-life situations such as:

> Ten persons need 4 kg of sugar per week. How much sugar do a group of 50 persons need for a 14 day holiday camp?

A detailed account on proportion problems is, for example, given by Vergnaud [5].

It is important to stress here that a target of the instruction on proportionality should be the concept of function not the rule of three (the traditional solution of proportionality tasks; an able, yet somewhat outdated relic from the past times). To achieve this end, students should first realize relations such as "If a increases 3 times, b will also do so" and "If c increases 5 times, d will decrease 5 times", gradually conceptualising appropriate functional forms of direct and indirect proportionality.

Let us finally illustrate a transition from proportionality to function by using the concept of uniform motion, which requires that paths of equal lengths are covered in the same time intervals. This concept can be defined by the equivalence of internal ratios

$$d_1 : d_2 = t_1 : t_2$$

as well by the constancy of the external ratio

$$d : t = \text{const},$$

which is obtained from the equivalence of external ratios

$$d_1 : t_1 = d_2 : t_2.$$

However, as Freudenthal [2] reminded us, this cognitive leap from the internal to the external definition of uniform motion, which is based upon interchanging the middle, goes almost unnoticed by those who are familiar with proportions. (We should not forget that the Greek tradition only accepted ratios between magnitudes of the same kind.) By generalizing this discussion, the concept of linear function can be defined by the equivalence of internal ratios

$$y_1 : y_2 = x_1 : x_2,$$

as well as by the constancy of the external ratio

$$y : x = \text{const}, \text{ or } y = kx, \quad k = \text{const},$$

which may be regarded—respectively—as implicit, conceptual definition and explicit procedural definition of linear function.

Methodology

Subjects

The study used a sample of 68 Gymnasium (high-school) students from two ninth-grade classes. The subjects' average age was 15 years and 47% of them were male. According to an entrance examination test for the upper secondary education (grades 9–12), the subjects' mathematical abilities were average. The subjects were taught mathematics by the author of this study, the existence of which was completely unknown to them.

Design

The study had a pre-test/post-test design with two parallel groups. The variables were: treatment condition, initial quantitative reasoning, initial qualitative reasoning, final quantitative reasoning, final qualitative reasoning, gain in quantitative reasoning (the difference between final and initial quantitative reasoning), gain in qualitative reasoning (the difference between final and initial qualitative reasoning), the coordination of initial quantitative and qualitative reasoning, and the coordination of final quantitative and qualitative reasoning.

Instruments

The pre-test comprised the following two items assessing the subjects' quantitative and qualitative reasoning, respectively:

1. Five workers plant 120 seedlings in 8 hours.
   a) in 5 hours 4 workers will plant ____ seedlings;
   b) 210 seedlings will be planted in 7 hours by ____ workers;
   c) 3 workers will plant 90 seedling in ____ hours.
2. A number of campers consume a certain amount of sugar in some days.
   a) longer camping with less amount of sugar involves:
   less campers / more campers / cannot be answered
   b) shorter camping with more campers requires:
   less sugar / more sugar / cannot be answered
   c) camping with less campers and more sugar lasts:
   shorter / longer / cannot be answered

   Underline the correct answers and give short explanations without using numerical data.

Its alpha reliability obtained from the subjects' scores was .72 (.89 for the first three questions and -.08 for the others as 93% of the subjects scored 0 on them). A hierarchical cluster analysis did evidence two type of questions (questions 1.a–c vs. questions 2.a–c).

The post-test comprised the following two items assessing the subjects' quantitative and qualitative reasoning, respectively:

1. Ten campers consume 3.5 kg of sugar in 7 days.
   a) 5 kg of sugar will be consumed in 4 days by ____ campers;
   b) 6 campers will consume 3 kg of sugar in ____ days;
   c) in 5 days 4 campers will consume ____ of sugar.
2. A number of workers plant a certain number of seedlings in a number of hours.
   a) planting less seedlings with more workers is realized in:
      shorter time / longer time / cannot be answered
   b) planting more seedlings in shorter time requires:
      less workers / more workers / cannot be answered
   c) less workers in longer time plant:
      less seedlings / more seedlings / cannot be answered

   Underline the correct answers and give short explanations without using numerical data.

Its alpha reliability obtained from the subjects' scores was .80 (.93 for the first three questions and.81 for the others). A hierarchical cluster analysis also evidenced two type of questions (questions 1.a–c vs. questions 2.a–c). Note that the pre-test and post-test scores were correlated (.32, $p < .01$).

Procedure

The pre-test was administered on Tuesday. The test was followed by a 45-minute treatment on solving multiple proportion problems that was realized on Wednesday. The post-test was administered on Thursday. Both tests were administered in a group setting during regular mathematical lessons. The instruments were scored by the author. Only correct solutions demonstrating sound reasoning within the underlying model (e.g., $z = Cx/y$, $C = \mathrm{const}$) were accepted.

Treatment

The treatment, the main objective of which was to externalise the underlying model of the examined tasks, was realized by the author during two regular lessons.

- The QN group ($N = 33$) exclusively solved quantitative problems regarding questions 1.a–c of the pre-test. The questions were initially answered arithmetically by finding out the number of seedlings planted in one hour by one worker. This solution strategy was then algebraically described by the equation $A = x/(yz)$ ($A = \mathrm{const}$), and the same questions were answered again by using it and its equivalent forms. The students were then asked "Where is direct and/or inverse proportion in the applied equations?" Having recalled the known equations $y = kx$ and $y = k/x$ and having played with them (if x increases, y increase as well, etc.), they gradually realized facts such as "more workers work shorter" and "less workers plant less seedlings". However, nothing was said about how simultaneous changes in two variables affect the third one. Qualitative questions 2.a–c of the pre-test were given for homework.

- The QL group ($N = 35$) only solved qualitative problems regarding questions 2.a–c of the pre-test. The questions were answered through filling a table regarding all possible situations (e.g., if the number of campers increases and the camp duration decreases, the direction of the change in the amount of sugar cannot be qualitatively determined). Sixteen situations were carefully examined (questions 2.a and 2.b in all possible variants: $2 \cdot (3 \cdot 3 - 1)$). Having recalled the known equations $y = kx$ and $y = k/x$ and having played with them (if x increases, y increase as well, etc.), we described these situations by two underlying equations. The analysis and formalization of the remaining eight situations concerning question 2.c was left for homework. Quantitative questions 1.a–c of the pre-test were also given for homework with the suggestion "to solve them, find the constant". However, any suggestion regarding how to do it and how to solve the proposed task was not given.

Note that the main objective of the treatment was clearly explained to the subjects and many of them actively participated in problem solving.

Statistical analysis

As the collected data mostly did not come from normal distributions, the following statistical analysis was applied:

- the difference between the treatment groups relating to the reasoning variables was assessed by the Mann-Whitney U test—a nonparametric version of the *t*-test for independent samples;
- for each treatment group, the difference between the initial and final corresponding variables was tested by means of Wilcoxon Matched-Pairs Signed-Ranks Test—a non parametric version of the *t*-test for paired samples;
- the coordination were measured by the Spearman correlation coefficient—a nonparametric version of the Pearson correlation coefficient.

Results

The mean percentage of correct responses regarding the reasoning variables for the treatment groups are reported in Table 1. Despite numerical differences, the results of the treatment groups were statistically equal in respect to the initial quantitative reasoning ($U = 508.5$, $Z = -.97$, $p = .33$), the initial qualitative reasoning ($U = 563.0$, $Z = -.39$, $p = .69$), the final quantitative reasoning ($U = 498.0$, $Z = -1.26$, $p = .21$), and the gain in quantitative reasoning ($U = 547.5$, $Z = -.41$, $p = .68$). The group QL outperformed the QN group regarding the final qualitative reasoning ($U = 423.5$, $Z = -2.06$, $p < .05$). Furthermore, the QL group outperformed the QN group in respect of the gain in qualitative reasoning ($U = 429.0$, $Z = -1.96$, $p < .05$). According to the Wilcoxon Matched-Pairs Signed-Ranks Test, the subjects from both treatment groups *only* improved their qualitative reasoning ($Z = -2.76$, $p < .01$ – the QN group; $Z = -4.06$, $p < .01$ – the QL group).

VARIABLE	QN	QL
initial quantitative reasoning	35	24
initial qualitative reasoning	**2**	**3**
final quantitative reasoning	29	17
final qualitative reasoning	**24**	**41**
gain in quantitative reasoning	−6	−7
gain in qualitative reasoning	**22**	**38**

Table 1. Mean percentage of correct responses regarding the reasoning variables for the treatment groups

Table 2 presents the coordination of quantitative and qualitative reasoning for the test types and the treatment groups. The coordination between final quantitative reasoning and final qualitative reasoning was significant for the QL group. [For this group, the partial Pearson correlation between final quantitative reasoning and final qualitative reasoning when initial quantitative reasoning and initial qualitative reasoning were controlled was .40 ($df = 31$, $p = .02$)].

TEST TYPE	QN	QL
pre-test	.08	.29
post-test	.18	$.41^*$

* $p < .05$

Table 2. Coordination of quantitative and qualitative reasoning for the test types and the treatment groups

Discussion

The study showed that: (a) the examined problems were hard for most subjects; (b) even in the QN group, quantitative reasoning was not improved; (c) qualitative reasoning was improved in both treatment groups and the QL group scored better; and (d) only the QL treatment related quantitative and qualitative reasoning.

The post-test evidenced that, on average, the subjects' success rate was 28%. (1.7 out of 6 correct answers). As only correct solutions demonstrating sound reasoning within the examined underlying model were accepted, 29 subjects (43% of the sample!) scored zero on the post-test. These students were not able to find the constant and use it efficiently, even after the QN treatment (17 out of 33 subjects). We do not believe that a 60 minute treatment might result in much better outcomes. (It is indeed difficult to defend a longer treatment for average mathematics students that cultivates only one type of reasoning.) Note that according to Vergnaud [5], only about 60% of students from a tenth-grade sample could solve quantitative multiple proportion problems regarding consumption.

Contrary to qualitative reasoning, quantitative reasoning was not improved in each of the treatment groups. Furthermore, the initial and final quantitative reasoning simply did not correlate in each of the treatment groups. As the same pattern was observed for the initial and final qualitative reasoning, we hypothesized that most subjects probably viewed the test tasks as four distinct problems instead of the instances of two distinct problems. It might indeed be somewhat true since a 12-item hierarchical cluster analysis yielded two supportive solutions: a 4-cluster solution (questions 1.a–c of the pre-test; questions 2.a–c of the pre-test and question 2.c of the post-test; questions 1.a–c of the post-test; and questions 2.a–b of the post-test) and a 3-cluster solution (questions 1.a–c of the pre-test; questions 2.a–c of the pre-test, question 2.c of the post-test and questions 1.a–c of the post-test; and questions 2.a–b of the post-test). This important finding clearly suggests that problem solving should also require solvers to generate contextually different problems having the same underlying structure, without paying particular attention on concrete numerical data. This "context-play" activity was unfortunately missing in our treatment since we wrongly believed that our subjects could themselves conceive that the chosen context (work or consumption) does not affect the underlying structure and its model.

A study of Harel & Behr [3] proposes two important questions: "Can students learn principles for qualitative reasoning on proportion problems?" and "Can skilful qualitative reasoning facilitate quantitative reasoning in this area?". Having in mind the QL treatment outcomes, the study answers affirmatively to the first question. However, despite linking quantitative and qualitative reasoning, it evidences that skilful qualitative reasoning does not necessarily imply competent quantitative reasoning, which is in accord with Behr et al. [1].

To summarize, this study, like that of Mayer, Lewis & Hegarty [4], evidences that qualitative reasoning skills should be cultivated in addition to traditionally developed quantitative reasoning skills. Furthermore, it evidences that a qualitatively-oriented teaching may be more efficient than a quantitatively oriented one. As the QL treatment did not promote a gain in quantitative reasoning, further studies may examine how these kinds of teaching involving the "context-play" activity should be sequenced (or combined) in order to promote both the acquisition and coordination of quantitative and qualitative reasoning.

REFERENCES

[1] Behr, M. J., Harel, G., Post, T. & Lesh, R., *Rational number, ratio and proportion*, In: Grouws, D. A. (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, (pp. 296–333), New York: Macmillan, 1992.

[2] Freudenthal, H., *Weeding and Sowing*, Dordrecht, Holland: D. Reidel Publishing Company, 1978.

[3] Harel, G. & Behr, M., *The blocks task: expert solution models*, Journal of Structural Learning, **11**, 2 (1992), 173–188.

[4] Mayer, R.E., Lewis, A.B. & Hegarty, M., *Mathematical misunderstandings: qualitative reasoning about quantitative problems*, In: Campbell, J.I.D. (Ed.), *The Nature and Origins of Mathematical Skills* (pp. 137-153), Amsterdam: North-Holland, 1992.

[5] Vergnaud, G., *Multiplicative structures*, In: Hiebert, J. & Behr, M. (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 141–161), Reston, VA: National Council of Teachers of Mathematics, 1989.

Djordje Kadijevich, Megatrend University of Applied Sciences & Mathematical Institute SANU, Belgrade, Serbia & Montenegro

E-mail: djkadijevic@megatrend-edu.net, djkadij@mi.sanu.ac.yu