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USING INNOVATIVE TECHNOLOGY FOR REVITALIZING FORMAL AND INFORMAL MATHEMATICS: A SPECIAL VIEW ON THE INTERPLAY BETWEEN PROCEDURAL AND CONCEPTUAL KNOWLEDGE

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Abstract. A student often meets a conflict between conceptual and procedural knowledge: does (s)he need to understand for being able to do, or vice versa? Hence an important research question is how pedagogical solutions affect the relation between the two knowledge types. Our theoretical analysis and practical experience evidence that desired links can be promoted when the learner has opportunities to simultaneously activate conceptual and procedural features of the topic at hand. Such activation is considered for interactive learning that utilizes an able technological tool, the *ClassPad* calculator. In a sequence of examples, we will show how this tool can be exploited to develop both informal and formal mathematical knowledge.

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Logical basis of conceptual and procedural knowledge

Having made a thorough analysis concerning the studies on conceptual and procedural mathematical knowledge, we [5] proposed the following knowledge characterization respecting modern theories of teaching and learning.

- *Procedural knowledge* (or **P** hereafter) denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation forms. This usually requires not only the knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.
- *Conceptual knowledge* (or **C** hereafter) denotes knowledge of and a skilful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may intro- duce a new concept or rule) given in various representation forms.

It is especially the dynamic and semantic view of **C**, which we wanted to highlight more clearly. Furthermore, the two knowledge types can, in some cases,

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be distinguished only by the level of consciousness of the applied actions. While **P** often calls for automated and unconscious steps, **C** typically requires conscious thinking. However, **P** may also be demonstrated in a reflective mode of thinking when, for example, the student skillfully combines two rules without knowing why they work.

Concerning links between **P** and **C** (P-C links hereafter), we found that four relations could be extrapolated from relevant research studies. These are:

- *Inactivation view* (I): **P** and **C** are not related [12], [13].
- *Simultaneous activation view* (SA): **P** is a necessary and sufficient condition for **C** [7], [1], [3].
- *Dynamic Interaction view* (DI): **C** is a necessary but not sufficient condition for **P** [1].
- *Genetic view* (G): **P** is a necessary but not sufficient condition for **C** [10], [9], [2], [16], [14].

Having in mind different student abilities, various teaching approaches and diverse topics with associated learning problems, it is appropriate to stress that these views do not evidence any general conclusion regarding the relation between **P** and **C**.

Dynamic interaction and simultaneous activation

Because of the dominance of **P** over **C** in the development of scientific and individual knowledge, a reasonable pedagogical idea could be to go for spontaneous **P**, hoping that an appropriate **C** would be attainable, finally. On the other hand, it seems appropriate to claim that the goal of any education should be to invest on **C** from the first beginning. The SA method combines both of these demands in a natural way. However, it is the pedagogical framework that matters when planning how to promote P-C links in a learning environment. In [5, pp. 147-153] we summarized two pedagogical approaches:

- *Educational approach* is based on the assumption that **P** depends on **C**. Thus, the logical P-C links background is the DI or SA view. The adjective "educational" refers to educational needs, typically requiring a large body of knowledge to be transferred and understood.
- *Developmental approach* assumes that **P** enables **C** development. The logical P-C links background is the G or SA view, and the adjective "developmental" reflects the philogenetic and ontogenetic nature of mathematical knowledge.

The interplay of these approaches can be illustrated in a flourishing way only if the framework theory of knowing and learning is linked to the considerations. In this paper we will just give some ideas how progressive educational tools enabling DI and SA can be utilized. We therefore ask the reader to accept just a short verbal description of how the educational approach can be the leading framework, and how the developmental approach can be used to trigger the learning process. Figure 1 summarizes such a constructivist framework, having been presented in detail by [3], [4] and applied in a computer-based learning program for the conceptual field *Proportionality – Linear Dependence – Gradient of a Straight Line through Origin*. The program is freely downloadable from `http://www.joensuu.fi/lenni/programs.html`.

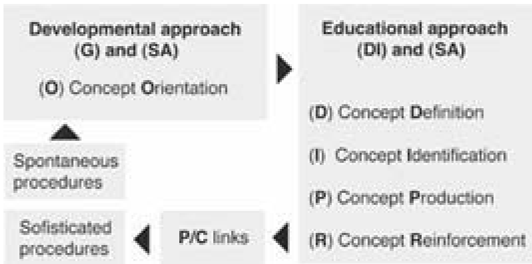

Fig. 1. A sophisticated interplay of the developmental and educational approaches

As regards the above-mentioned conceptual field, it seems appropriate to start with a spontaneous **P** and restrict the construction space by simplifying **C**: gradient is considered as a concrete slope, at first. Pupils can handle learning situations like one in Figure 2 by using spontaneous **P** based on their experiences without any explicit thinking of the mathematical relations between the objects[1]. This kind of *Orientation* (the first phase of the *concept building*) basically utilizes developmental approach: the interpretations are based on pupils' mental models and more or less naive procedural ideas.

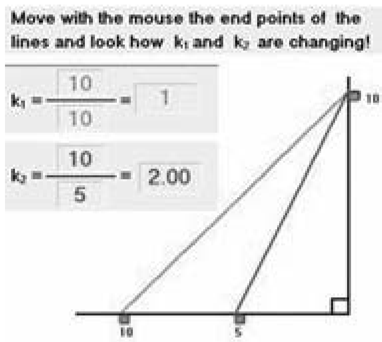

Fig. 2. Building a bridge between **P** and **C** manipulating parallel segments

These act like a wake-up voltage in an electric circuit that triggers another, more powerful current to be amplified again. **P** and **C** start to accelerate each other, offering a nice opportunity to use SA, for example. Technology allows students opportunities to manipulate the concrete slope visually and look how its abstract symbolic representation is changing. The mental constructions done by the student do not need to begin from the concrete or abstract, but from the relation between abstract and concrete, and even between abstract things. In the above-mentioned

[1]Figure 2 represents a solution made by an expert learner. However, in school and service-in-training we have met only novice learners. Pupils' and teachers' poor abilities to regulate own learning processes seem to prevent constructing links between concrete and abstract. They cannot simplify the situation by changing just one component (either the height or the breadth, and even that of one particular segment) at a time, and by looking, which components in the symbolic representations would change. Instead of that, they insist in changing all things at the same time, getting a data overflow [4, p. 10]

learning program, the user can find examples how to move from the concrete slope to the abstract mathematical concept *gradient* by utilizing the SA method again, and how DI method is involved in the other phases of *concept building* (*Definition, Identification, Production and Reinforcement*).

Utilizing SA method with ClassPad

For about 20 years, it has been possible to interpret symbolic representations as graphs by using home computers. Paradoxically, students should learn to understand these symbolic representations first before being able to utilize computers in this conventional way. We would like to illustrate SA activities by utilizing *ClassPad* 300—a modern pocket computer made by Casio (see `http://www.classpad.org/Classpad/Casio_Classpad_300.htm`).

Most *ClassPad* applications support a simultaneous display of two windows, allowing the user to access the windows of other applications from the main application and to perform drag-and-drop activities (i.e. copy and paste actions) and other operations with expressions between the *Main Application* work area and the currently displayed screen (*Graph Editor, Graph, Conic Editor, Table, Sequence Editor, Geometry, 3D Graph Editor 3D Graph, Statistics, List Editor* and *Numeric Solver*).

Let us start with two simple examples, which show how the properties of dynamical geometry programs have been extended to allow interplay between algebra and geometry.

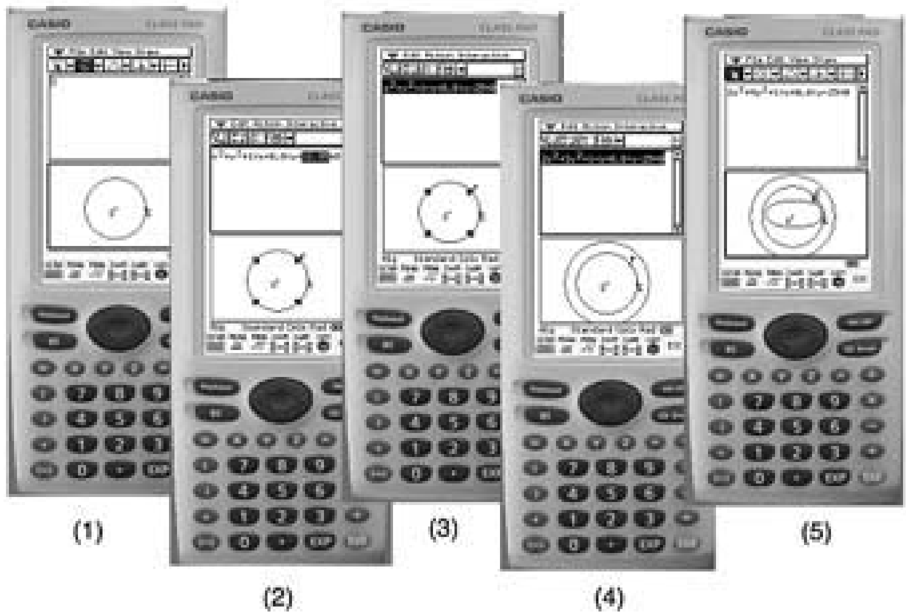

Fig. 3. Making links between algebra and geometry with *ClassPad*

Example 1. Without knowing anything about the analytic expression of a circle, we can just play harmlessly by drawing a circle in the geometry window (Fig. 3(1)), and then drag-and-drop the circle into the algebraic window (2). Something surprising happens: The circle seems to be expressed in algebraic form $x^2 + y^2 + 0.8xy - 12.55 = 0$. Let us manipulate (3) the equation by changing 12.55 to 25, then drag-and-drop it to see the new circle (4). It seems that only the radius changes. Let us go back to the algebraic window to do more manipulations (5). This time, let us change the coefficients of the second degree variables: $2x^2 + 9y^2 + 0.8xy - 12.55 = 0$ seems to make an ellipse. Of course a co-ordinate system can be added to geometry window, and the manipulation of the circle (or other curve) can be done in a more dynamical way by moving or stretching it within this window.

Anticipating that some readers might question this kind of informal mathematics, we would like to point out that the aim of the used SA method here has been to enhance mental links made by the student and not to produce any exact mathematics, yet. Of course, *ClassPad* modules would allow us to continue the above analysis on a more exact level by using plotting options as 'Sketch' or 'Conics'. The table below shows other types of expressions you can drag-and-drop between the 'Main Application' and the 'Geometry' window.

Table 1. Drag-and-drop *ClassPad* objects

Main Application window	Geometry window
linear equation in x and y	An infinite line
equation of circle in x and y	A circle
2-dimensional vector	A point or a vector
2×2 matrix	A transformation
equation $y = f(x)$	A curve
$n \times 2$ matrix	A polygon (each column represents a vertex)

Example 2. Figure 4 illustrates, how *ClassPad* can be utilized for orientation to the above- mentioned simplified conceptual field (gradient as concrete slope). Let us start by drawing a horizontal line (1). Drag-dropping it to the other window produces equation $y = 0$. Changing 0 to 2 produces the parallel line in (2). Playing with rotation tool by choosing the angle 45° produces a sloping line (3–4), which, after drag-dropping, appears to be represented by equation $y = x + 2.5$, and changing 2.5 to 5 seems to keep the parallelism (5), whereas changing the coefficient of x to 2 changes the slope (6). An interesting thing happens when the line is rotated by 90°: its algebraic expression changes from $2x$ to $-0.5x$, which suggests a hypothesis to be tested later on.

Example 3. Let us draw in the Geometry window (Fig. 5 (1)) a vector from origin to the point $(2, 1)$. A drag-and-drop activity produces its presentation as a matrix (2). Hypothesis: A vector seems to be equivalent to a transition from point A to point B. Now we can manipulate the vector and see what happens: Moving it does not seem to have any effect to its presentation (3), whilst a *Translation* (4) seems to have something to do with a new matrix again (5): the starting point of

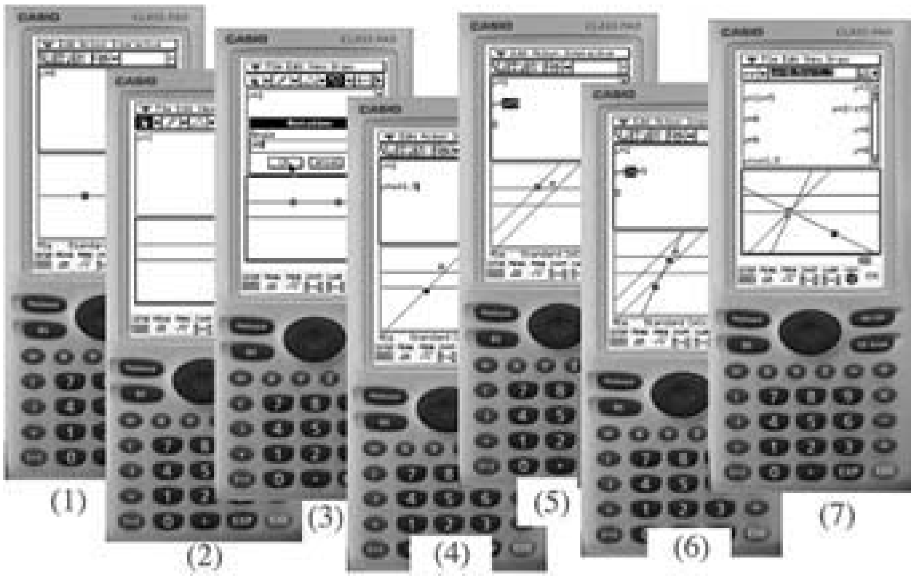

Fig. 4. Utilizing *ClassPad* for orientation to the simplified conceptual field

the vector seems to move as we just made our hypothesis concerning what a vector actually seems to be. Now our appetite is increasing: we find a new operation called *General Transformation*. The impact of the latter matrix seems to have the same effect, which we just tested (not illustrated in the figure). What about the more complicated matrix in (6)? By manipulating it we can make a hypothesis: "A general transformation seems to consist of a rotation and a translation, both being representable by a matrix".

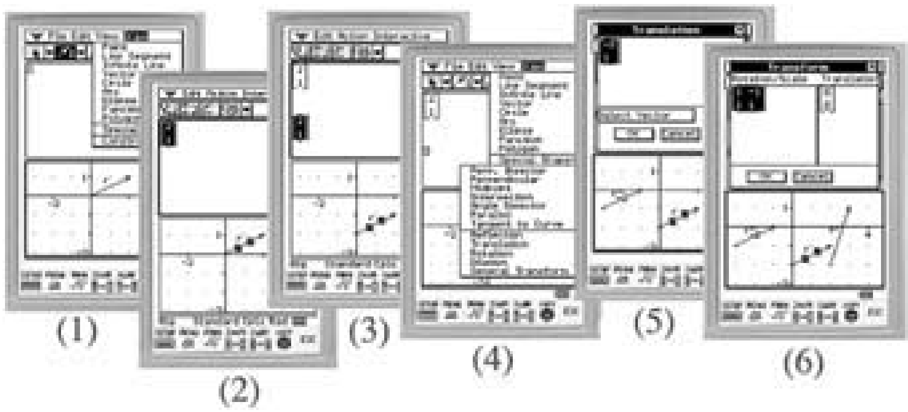

Fig. 5. Utilizing *ClassPad* for orientation to vectors and general transformations

Example 4. Hypotheses concerning a new operation *Diff* can be made through the following process: After drawing the curve of $x^3 - x$ in (1) of Figure 6, *Diff* produces a new expression $3x^2 - 1$, which can be drag-dropped into the same window as a new curve (3). Continuing the process, a drag-dropping of 6x produces a line into the same window (4). Now various kinds of manipulations can be made for all three curves for finding out what is the link between a curve and the matching "Diff-curve". If we move the original curve vertically as in (5), for example, the other two curves do not change at all. We can continue algebraic or geometric manipulations and utilize the small expression window for that purpose. Of course, only imagination puts limits what other operations and menus of *ClassPad* could be activated for enlarging the conception what *Diff* actually is!

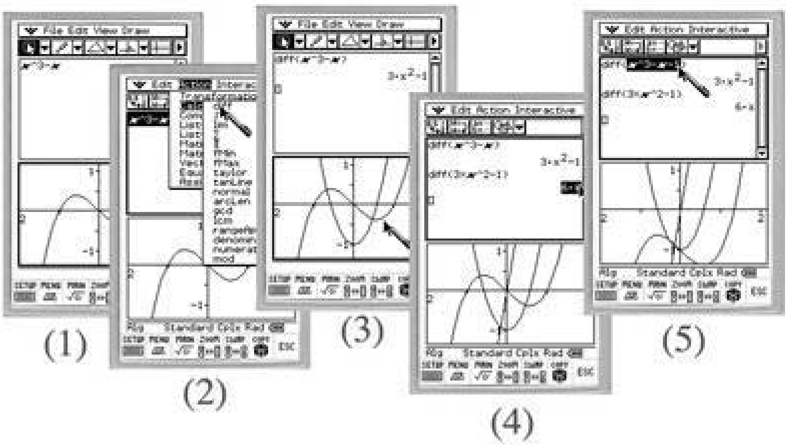

Fig. 6. Making hypotheses concerning derivative.

Closing words

For too many teachers and students within traditional institutions, mathematics consists of a number of discrete courses, which are frequently studied with little interdependence. It is not unusual, even for a student at university, to complete the undergraduate studies of the subject called 'mathematics' and yet to have little idea of what 'the subject' really is except the name of a set of its parts. Institutions have tended to take the view that, at any given level, there is a set of basic skills and concepts, which must be learned and practiced before any engagement can be attempted with the actual practice of mathematics. Within that tradition there is also a widespread single style of teaching and learning. Mathematics tends to be explained as an organized body of knowledge, in which students are largely passive, practicing old, clearly formulated, and unambiguous questions for timed examinations. The large body of theory is found to be abstract and depends on an unfamiliar language. These features are of course essential for the purposes of a professional mathematician, but they leave many students dispirited and bored, and their performance in more advanced courses is poor because the foundations are weak: the examiners are reduced to setting only bookwork or stereotyped questions, which can be remembered without becoming a vital part of the student.

Beside presenting the logical organization of mathematical knowledge, the focus should be on developing the student's ability to construct and understand knowledge instead of merely collecting data. Both technological applications and the history of mathematics offer excellent problem fields—already in elementary mathematics—for developing students' high-level skills from the very beginning. Understanding the technological and cultural perspectives of mathematics is to our mind an effective preventative measure against students' negative beliefs about mathematics, poor self- confidence and inter-cultural contradictions.

We cannot make any definitive conclusions about how, even less in which order, students' knowledge develops in each situation and for each topic. Even the most abstract concepts can be based on their spontaneous ideas. This, however, does not predestine any order for learning activities, because it is the pedagogical framework that matters. Our position is that doing mathematics should be cognitively meaningful for the student. Building a bridge between geometry and algebra, or concrete and abstract, is just one opportunity to utilize simultaneous activation by a *ClassPad* type of technology. We believe that it is a very promising step towards revitalizing the making of mathematics even on students' free time. A detailed analysis of TIMSS and PISA results reveal [11], [15] that it is not necessarily the school teaching that promotes students' mathematical knowledge. This makes educational research more interesting generating the following question: "Which factors in education are important for the development of thinking abilities?" Implementing of a new technology makes pedagogical issues complicated but on the other hand opens room for alternative teaching and learning paths, especially when learning through design is utilized [6]. Our task is to uncover and explore these paths contributing to a better mathematics education for both students and their teachers. If we accept the assumption that the main task of education is to promote a skilful 'drive' along knowledge networks so as to scaffold pupils to utilize their rich activities outside school, it seems appropriate to speak about an educational approach in the sense of this paper.

REFERENCES

1. Byrnes, J. & Wasik, B., *Role of Conceptual Knowledge in Mathematical Procedural Learning*, Developmental Psychology 27 (1991), 777–787.
2. Gray, E. & Tall, D., *Success and Failure in Mathematics: The Flexible Meaning of Symbols as Process and Concept*, Mathematics Teaching 142 (1993), 6–10.
3. Haapasalo, L., *Planning and Assessment of Construction Processes in Collaborative Learning*, in: S. Jarvela & E. Kunelius (eds.), *Learning & Technology—Dimensions to Learning Processes in Different Learning Environments* (1997), 51–66, `http://herkules.oulu.fi/isbn9514248104`
4. Haapasalo, L., *The Conflict between Conceptual and Procedural Knowledge: Should We Need to Understand in Order to Be Able to Do, or vice versa?*, in: L. Haapasalo & K. Sormunen (eds.), *Towards Meaningful Mathematics and Science Education*, Proceedings on the 19th Symposium of the Finnish Mathematics and Science Education Research Association, University of Joensuu, Bulletins of the Faculty of Education 86 (2003), 1–20.
5. Haapasalo, L. & Kadijevich, Dj., *Two Types of Mathematical Knowledge and Their Relation*, Journal für Mathematik-Didaktik 21 (2) (2000), 139–157.
6. Haapasalo, L. & Kadijevich, Dj., *Simultaneous Activation of Conceptual and Procedural Mathematical Knowledge by Means of ClassPad*, Proceedings of the ITEM2003 Conference in Reims, France (2003), `www.reims.iufm.fr/Recherche/ereca/colloques/`

7. Hiebert, J. (ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*, Hillsdale: Erlbaum (1986).
8. Kadijevich, Dj. & Haapasalo, L., *Mathematics Teachers as Multimedia Lessons Designers*, Proceedings of the ITEM2003 Conference in Reims, France (2003), www.reims.iufm.fr/Recherche/ereca/colloques/
9. Kitcher, P., *The nature of Mathematical Knowledge*, NY: Oxford University Press, 1983.
10. Kline, M., *Mathematics: The Loss of Certainty*, NY: Oxford University Press, 1980.
11. Kupari, P., *Finnish Students' Mathematical Literacy in the PISA 2000 Study*, in: L. Haapasalo & K. Sormunen (eds.), *Towards Meaningful Mathematics and Science Education*, Proceedings on the 19th Symposium of the Finnish Mathematics and Science Education Research Association, University of Joensuu. Bulletins of the Faculty of Education 86 (2003), 83–90.
12. Nesher, P., *Are Mathematical Understanding and Algorithmic Performance Related?*, For the Learning of Mathematics 6 (1986), 2–9.
13. Resnick, L. & Omanson, S., *Learning to Understand Arithmetic*, in: R. Glaser (ed.), *Advances in Instructional Psychology (Vol. 3)*, Hillsdale, NJ: Erlbaum, 1987.
14. Sfard, A., *Reification as the Birth of Metaphor*, For the Learning of Mathematics 14 (1994), 44–55.
15. Tornroos, J., *Finnish Mathematics Results in TIMMS 1999—a More Detailed Analysis*, in: L. Haapasalo & K. Sormunen (eds.), *Towards Meaningful Mathematics and Science Education. Proceedings on the 19th Symposium of the Finnish Mathematics and Science Education Research Association*, University of Joensuu, Bulletins of the Faculty of Education 86 (2003), 64–73.
16. Vergnaud, G., *Epistemology and Psychology of Mathematics Education*, in: P. Nesher & J. Kilpatrick (eds.), *Mathematics and Cognition: A Research Synthesis by the International Group for PME*, Cambridge: Cambridge University Press, 1990.

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