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A BROADER WAY THROUGH THEMAS OF ELEMENTARY SCHOOL MATHEMATICS, VI

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Abstract. Multiplicative structure of the block of numbers 1 – 100 is designed here as an activity of comprehension of multiplicative (and divisional) schemes, followed by their symbolic coding. At this stage, arithmetic operations and their properties gain their meaning as pull-outs from intuition. To help grasp that meaning and recognition of outer space appearances, we are elaborating here a system of iconic signs upon which such a learning process is founded. In particular, based on this system of signs, a method (that we call Pythagorean) for establishing properties of multiplication and division has been developed and discussed in depth here.

A step-by-step building of the multiplication table, iconic representation and explanation of its entries are elaborated as to facilitate learning and retention in memory.

At the end, two-box place holders have also been designed and digits in colour used for developing calculation skills in the case of addition and subtraction of two-digit numbers.

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Key words and phrases: Multiplicative structure of the block 1 – 100, Pythagorean method, multiplicative schemes.

11. Multiplication and division

The number block 1 – 100 is a natural frame for the establishment of meaning of multiplication and division as two mutually related operations. In both cases, the following main steps constitute the plan of elaboration of these operations:

(i) Comprehension of the multiplicative scheme (being divisional scheme as well),

(ii) Composition of the corresponding expressions (products in the former case and quotients in the latter),

(iii) Calculation of the numerical values of that expression (i.e. its decimalization).

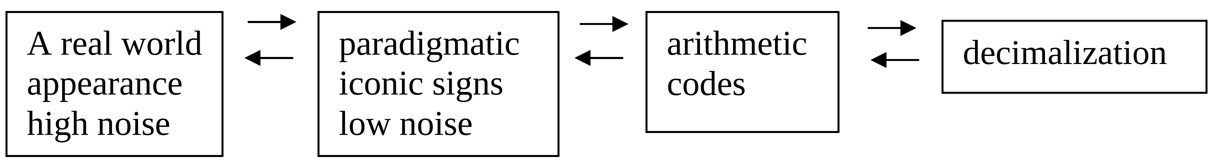
Besides the establishment of meaning of these operations, the block 1 – 100 is also the range of numbers where the multiplication table is built and upon which division (with or without remainder) is based. These number facts together with those in addition (subtraction) table form the complete fund of numerical facts that enter the child's long-term memory and which must be learnt

[1] To preserve the use of colours, this e-form of the paper is included and it differs from its hard-form only in number of pages.

systematically and through visualization. Since the so-called long calculations are reducible to the calculation with digits of the involved numbers, this empirical fund suffices in all cases of decimalization. Let us also add that the phrases as “mental multiplication”, etc. have no structural ground and they appear as being hopelessly old-fashioned.

Now we start to treat multiplication separately from division, concentrating first on the multiplicative schemes.

11.1. Following a general view. Not only in the case of elaboration of operations but generally, elaborating a whole range of related arithmetic topics, a plan of revealing and codifying structures is permanently present. Given schematically, that plan stands as follows



A real world appearance can be a spectacle in the outer space with inherent in itself mathematical structure or a pictogram representing that spectacle in a simplified way. A corresponding iconic sign is a pure shape conveying the same structure with the minimum of noise. Arithmetic codes are all sorts of realistic expressions and decimalisation is the process and the result of their tansforming which ends with decimal notation of their values.

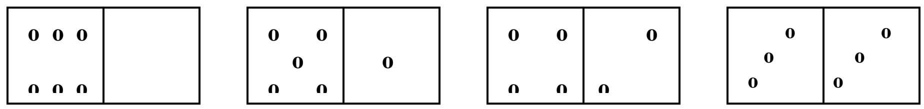
The arrows suggest links between components of the scheme, representing activities in both directions. In one direction they suggest the processes of abstracting and calculating and in the other one, reversed processes of returning to the original states and situations.

11.2. Multiplicative schemes. Comprehension of the multiplicative scheme consists of perception and recognition as two actions being inseperably interwined. When recognizing, the stimulus meterial, projected by a configurated structure, is fitted with the inner mental representations already existing in the mind of observer. On the other hand, in these first steps of elaboration, such representations have to be built and externalized in the form of graphical schemes. Thus, we face here a well-known dilemma „what comes firs, chicken or egg“. But this perplaxing situation happens to be already resolved in the spontaneous development of the child’s mind (and there is no need that we would analyze here adaptive reactions leading to that end). As a matter of fact, childern of this age understand easily the verbal description of the multiplicative scheme, which the old generations of teachers used to put in words, saying „7 places, in each 8 objects“, etc. Hence an appropriate elaboration should start with such examples:

(a) Two hands, on each five fingers. It is 2 · 5 fingers,

(b) Three cars, each four-wheeled. It is 3 · __ wheels,

(c)

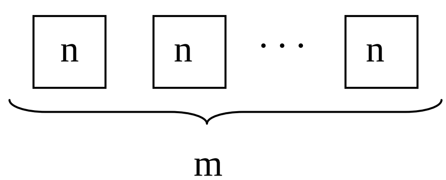


Four domino pieces, each having six dots. It is 4 · __ dots, etc.

At the beginning, the order of writing factors should be relevant. For instance, we can decide on the first factor to denote number of places and on the second, number of elements. Mathematically, this order is, of course, of no relevance and its role is to be just a cue having effects in the process of linking schemes to arithmetic codes.

Traditionally, there was an expectation that, on the basis of a conglomerate of suitable examples, a multiplicative scheme would be formed in the child's mind spontaneously. Gestalt approach is based on designs that bear essential resemblance to all their referents and from which, generic structural features can be grasped. Both, R. Skemp, [6] , and H. Freudenthal, [3], [17], use such designs, and we will, of course, follow that path. The domino pieces in example (c), form a configuration to which we can react either adding: 6 + 6 + 6 + 6 or multiplying: 4 · 6 and, therefore, it can be comprehended either as an additive or a multiplicative scheme. Generalizing that specific case, a multiplicative scheme can be described in several ways:

1. In words: m places, in each n elements.
2. As an iconic sign



(m boxes, in each n marbles).

3. Using the set theoretical language : disjoint union of m equipotent sets each having cardinality n.

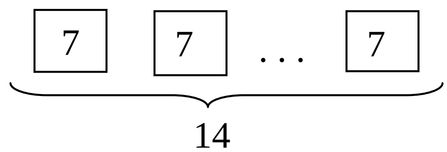
4. Enumerating disjoint sets by numbers 1,2, ... ,m and elements of each of them by numbers 1,2, ... ,n as the direct product

$$\{(i,j): i \leq m, j \leq n\} = \{1,2, ...m\} \text{ x } \{1.2, ... ,n\}.$$

(To each element, an ordered pair (i,j) is attached, i indicating the set to which the element belongs and j being the enumerating number of that element).

In a real teaching and learning situation only forms given under 1 and 2 are admissible. Here we will focus our attention on the iconic sign under 2 and we will call that form of multiplicative scheme „m boxes, in each n marbles“ scheme.

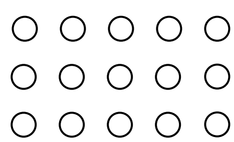
Quite clearly, in a didactical transposition, specific numbers should be used instead of m and n and the use of ellipsis as, for example, the following one



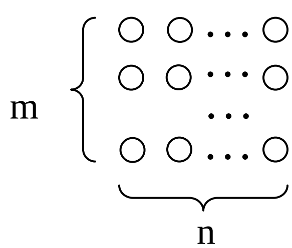
goes without any problems only if childern have understood the „and so on“ function of these dots.

Iconic signs are significant and their active and deliberate use is better than any verbal description of their function. Nevertheless, signs with ellipsis are hybrid. Iconically, they suggest the type of a configurated structure but their size is given in numbers (which are conventional symbols).

A still „nicer“ multiplicative scheme is obtained when equipotent sets are placed as rows of a rectangular scheme. For example, o’s are arranged in this scheme:



to form 3 rows and 5 columns. A general sign of this type is hybrid again:



with m and n indicating numbers of rows and columns respectively.

When rows of a rectangular scheme are enumarated by numbers 1,2, ... ,m and columns by 1,2, ... ,n, each of this elements is uniquely attached to a pair (i,j). Conceived mathematically such a scheme is the direct product

$$\{1,2, \ldots ,m\} \times \{1,2, \ldots ,n\},$$

That means that it is a conveyor of the same mathematical structure as the „m boxes, in each n marbles“ scheme is. This scheme will be called *rectangular multiplicative scheme.*

Comprehension, as a dynamical process, consists of fighting noise and fitting a configurated structure with its norm – corresponding scheme. How far such a structure stands close to or away from the norm depends, of course, on the amount of noise contaminating that structure. Thus, comparing two kinds of multiplicative schemes, the former is more general and fits easier with various configurations and the latter resembles more the idea of direct product and is more suitable for some other purposes.

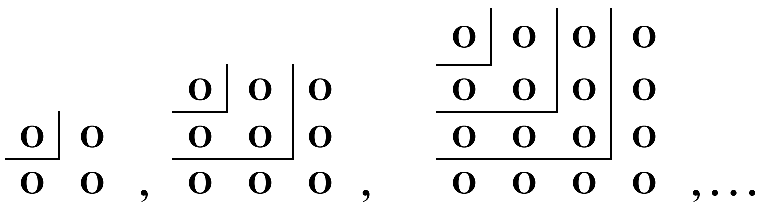
With more or less variation, rectangular schemes, as sorts of important spacial patterns, have their role in several other situations not necessarily related to multiplication. As an often present materialization of it we can take squared pages of a note book or the arrangement of pixels on a computer screen. Dotted rectangular schemes are also the ground for creation of interesting didactical games that can be assigned to childern of primary school age.

11. 3. Sorts of noise that matter. According to the Cantor's principle of invariance of number, through the process of observing a collection of objects and then, of ignoring their nature and any kind of their organization (ordering, grouping, arranging, etc.), we arrive at the pure idea of number. If for the abstract idea of number, organization of objects is unessential, for the way how such a number is comprehended and its arithmetic code composed, some specific ways of arranging and grouping objects are essential and, therefore, they constitute the sort of noise that matter. Appropriate examples are abundant: decimal grouping, groupings and arrangements to which we react adding, multiplying or, more generally, composing different kinds of arithmetic expressions. These sorts of „systematic“ noise, when present in a phenomenon – a configurated structure, a design used as an iconic sign, etc. can be conceived differently, leading so to different codes of the same number. Just that interplay of schemes and codes will be the subject of our consideration in the next subsection.

Let us observe that a difference should be made between grouping and arranging. The former is a decomposition of a set into its disjoint subsets and the latter is a spacial pattern giving the shape to a collection of objects.

11. 4. Pythagorean method – meaning-based acceptance of rules. In an actual didactical situation, the number block 1 – 100 is certainly not a right setting for establishment of all rules expressing propreties of multiplication. And then, in such a situation, particular cases and schemes bearing more resemblance to the surrounding realities should be employed. Leaving aside purely pedagogical questions of „what, when and how“, we now turn our attention to schemes as a means of further intuitive foundation of multiplication. Therefore, we will be using here general schemes and be considering general cases.

Going back to the history of mathematics, we recall that Pythagoras was using arrangements of dots to represent numbers and to establish relations between them. A famous example are the square-like arrangements of dots:



upon which equalities of sums of odd numbers and square numbers were established and what, in our contemporary notation, we write as the equality

$$1 + 3 + ... + 2n - 1 = n^2.$$

This method consists of two different ways of looking at the same arrangement and accordingly, of two different codings, which are then equated. We will call this procedure *Pythagorean method* and we will be using it here for establishment of arithmetic rules.

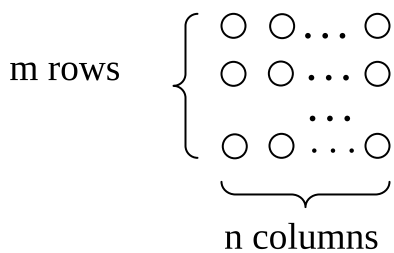
In Chapter 8 of his book [6], R. Skemp presents the way how some main rules related to multiplication are established on the basis of meaning , attached to suitable schemes, but „without calculating“. We will extend the number of such schemes and rules, and also include those rules which are applicable on transformation of expressions before the period, when deduction can be used and when a number of them are derived from a set of axioms.

Let us remark that, at this stage, the rules are principles on which, a phase in development of the acted-out arithmetic is based. Unfortunately, this fact does not seem to be widely appreciated and we can see these rules being „proved“ in some text-books, by calculating in a number of particular cases. And the Skemp’s „without calculating“ is the best warning sign for not doing that. Equally unfortunate is the phrase „we derive that ... “ , in a situation when all our derivations are pull-outs from intuition. Based on meaning, rules are acceptable principles (though some of them temporary) and, in the ultimate instance, they express some cues for our intelligent comprehension of the surrounding world.

11. 4. 1. Rectangular multiplicative scheme. Planted regularly in 8 rows, 12 seedlings grow. It is $8 \cdot 12$ seedlings. Looking at the same plantation from the side, 12 rows are seen, in each 8 seedlings. It is 12 · 8 seedlings. Since the number of seedlings does not depend on the way we see the plantation (or on the way how we group the seedlings in our thoughts), without calculating we write the equality 8 · 12 = 12 · 8.

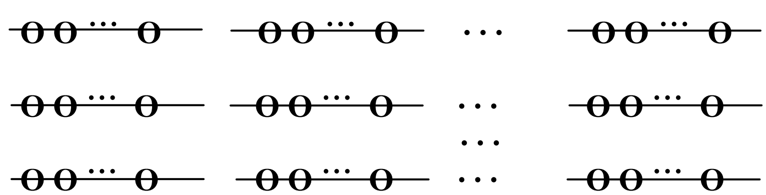
Continuing to do similar exercises, changing only the involved numbers, a feeling is formed that so obtained inequalities keep on holding independdently of the chosen numbers. Then, such a feeling makes us ready to accept the *rule of interchange of factors*.

In general, using the following arrangement of o's:



we see either m rows, in each n o's, what is altogether m · n o's or n columns, in each m o's, what is altogether n · m o's. Equating two denotations of the same number, the equality m · n = n · m follows.

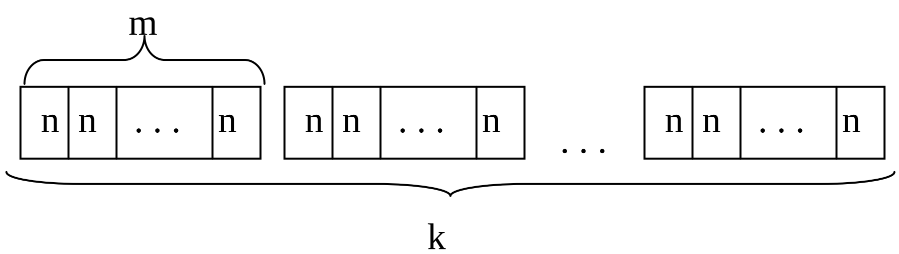
11. 4. 2. Multiplicatively multiplicative schemes. In k rows m strings, on each n pearls joined together, describes the following scheme



The number of strings is k · m and the number of all pearls is (k · m) · n. Or, m strings in a row, each having n pearls. It is m · n pearles in a row. Then, the number of all pearls is k · (m · n). Equating two denotations, the equality

(k · m) · n = k · (m · n) follows.

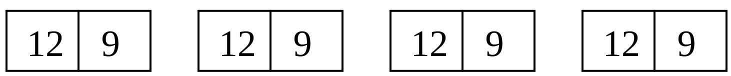
Another multiplicatively multiplicative scheme can be described as k packages of m boxes, in each box n marbles.



The number of boxes is $k \cdot m$ and the number of all marbles $(k \cdot m) \cdot n$. Or, in m boxes of a package there are $m \cdot n$ marbles. Then, the number of all marbles is $k \cdot (m \cdot n)$. Equating, $(k \cdot m) \cdot n = k \cdot (m \cdot n)$ follows again.

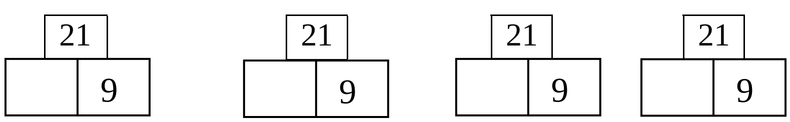
These two schemes are equally suitable for establishment of the *rule of association of factors*.

11. 4. 3. Additively multiplicative schemes. In each of 4 packages there are two boxes. In each package 12 blue marbles in the first box and 9 red in the second.



In each package 12 + 9 marbles. Altogether, it is $4 \cdot (12 + 9)$ marbles. Or, 4 boxes, in each 12 blue marbles, it is $4 \cdot 12$ blue marbles and 4 boxes, in each 9 red marbles, it is $4 \cdot 9$ red marbles. Altogether, it is $4 \cdot 12 + 4 \cdot 9$ marbels.
Equating, we get $4 \cdot (12 + 9) = 4 \cdot 12 + 4 \cdot 9$.

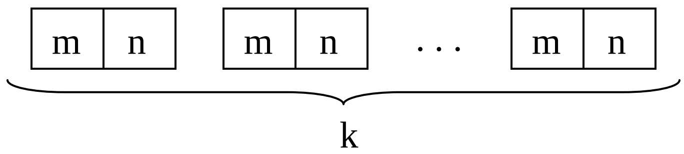
Having the same scheme but changing data: in each of 4 packages, there are 21 marbles, among them 9 red, we describe the following scheme



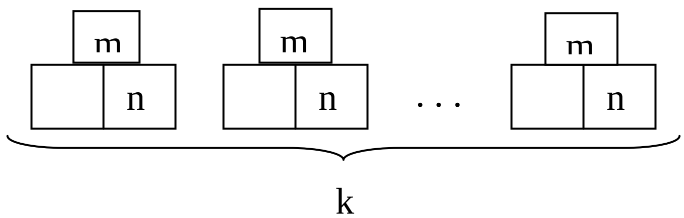
In each of 4 packages, there are 21 – 9 blue marbles. The total number of blue marbles is $4 \cdot (21 - 9)$. Or, in each of 4 packages, there are 21 marbles. It is $4 \cdot 21$ marbles. In each of 4 boxes, there are 9 red marbles. It is $4 \cdot 9$ red marbles. Then, the total number of blue marbles is $4 \cdot 21 - 4 \cdot 9$. Equating, we get $4 \cdot (21 - 9) = 4 \cdot 21 - 4 \cdot 9$.

Doing a number of similar examples, the *rule of multiplication of sums* and the *rule of multiplication of differences* are established and then, they can be expressed procedurally or rhetorically.

In general, the scheme

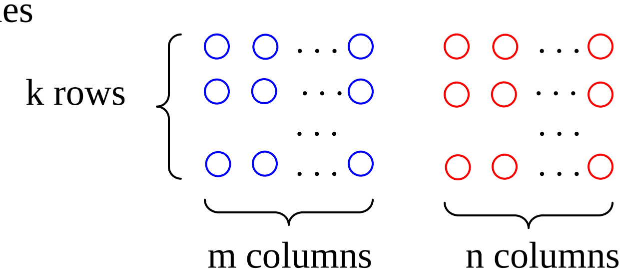


is used to establish k · (m + n) = k · m + k · n, and the scheme



to establish k · (m – n) = k · m – k · n.

Let us also add that the arrangement of k times m blue o's and k times n red ones



can be employed to serve the same purpose as well.

In summary, let us say that the schemes considered here are fundamental and that there still exists a variety of schemes giving the meaning to various arithmetic expressions and then serving us well for planing schematic learning.

11. 5. Gradual building of multiplication table. In the didactical building of the block 1 - 100, a step- by- step construction of the multiplication is its central topic. Contrary to the traditional learning by rote, a spontaneous way of acquiring the multiplication table will be sketched here. This spontanity does not mean, of course, dispersing off the table entries over the whole content of a text-book (as it is often seen) and waiting untill the last pupil in a class has them memorized (as it is often practised). What is meant here by a spontanious elaboration is a process of learning which is systematically planed, but carried out step by step over a longer period of time and without any forcing.

In the way how addition is superposed upon counting (and, in the cognitive hierarchy, is a higher order operation), it is exactly in the same way, that multiplication is superposed upon addition of equal summands. Reacting to the same scheme, as for example,

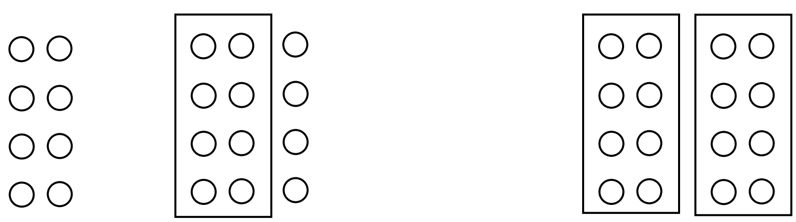
23	23	23

once we may add: 23 + 23 + 23 and the other time, multiply: $3 \cdot 23$. Without doubt, such two reactions relate these two operations and this relationship can be exploited in the early stage of calculation of some small products. However, what still survives as a least didactically poor procedure is the definition of multiplication as repeated addition of equal summands.

11. 5. 1. Multiplication table up to 4 4. All products: $2 \cdot 2$, $2 \cdot 3$, ... , $2 \cdot 9$ can be calculated as: 2 + 2, 3 + 3, ... , 9 + 9. For the sake of having a larger number of examples, when multiplication and addition are related, the products standing out of multiplication table as, for instance, $2 \cdot 10$, $2 \cdot 17$,
$2 \cdot 48$, ... could be also calculated reducing them to sums.

Using iconical representations as, for example,

$2 \cdot 3 = 6$, $3 \cdot 3 = 9$, (9 seen as 6 + 3)



$2 \cdot 4 = 8$, $3 \cdot 4 = 12$, (12 seen as 8 + 4), $4 \cdot 4 = 16$,(16 seen as 8 + 8)

the products completing the table up to 4 4 are introduced and spontaneously memorized through their use in a number of exercises.

11. 5. 2. Rearrangement explains multiplication table entries. A systematic building of the multiplication table starts with multiplying by 5, when structured groups of 5's are also seen as groups of 10's with or without one 5 remaining. Such groupings can be found in our paper „Shematic learning of addition and multiplication tables – sticks as concrete manipulatives“, this „Teaching“, vol. I, pp. 31 – 51, 1998 and here , we will not consider them again.

Let us also say that all groupings considered in that paper are designed to be seen, in the same time, as groups of n's, (n = 5, 6, 7, 8 and 9) and as tens with remaining units. As for multiples of 5, they are easy for memorization and we will not include them in this consideration.

In the case of multiplication by 6, 7, 8 and 9, we will consider rearrangements which, then, suggest the procedure of decimalization. For example, this means that an arrangement which suggests 7 groups of 9's, when rearranged, will suggest 6 groups of 10's and 3 units. Hence, such *rearranging explains the mening of entries in the multiplication table*. (To see why $7 \cdot 9 = 63$, groups of 9's are rearranged to form groups of tens and 3 units remain).

11. 5. 3. Multiplication by 6. Using the rule of multiplication of sums the products $2 \cdot 6$, $3 \cdot 6$, $4 \cdot 6$ and $6 \cdot 6$ are calculated this way

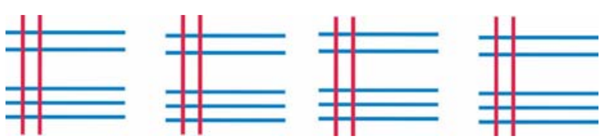
$$2 \cdot 6 = 2 \cdot (5 + 1) = 10 + 2 = 12, \quad 3 \cdot 6 = 3 \cdot (5 + 1) = 15 + 3 = 18, \ldots$$

hence reducing them to the multiples of 5.

Let us notice that such products can be calculated without an explicite use of this rule. Namely, $3 \cdot 6$ can be seen as the sum of 5's and1's:

and then, $3 \cdot 6 = 3 \cdot 5 + 3 \cdot 1 = 15 + 3 = 18.$

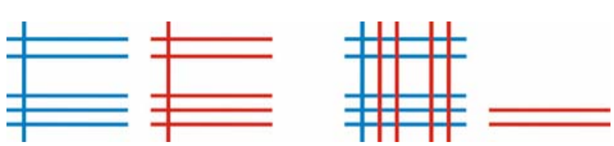
In the same way, for example, $4 \cdot 7$ is calculated as the sum of 5's and 2's



and then, we have $4 \cdot 7 = 4 \cdot 5 + 4 \cdot 2 = 20 + 8 = 28$.

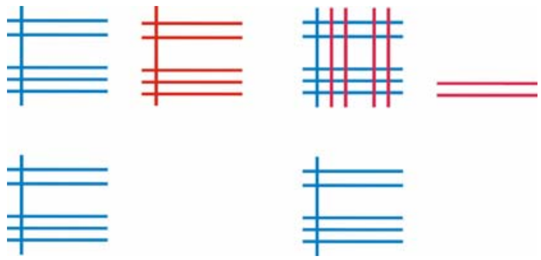
A weak aspect of such procedures is the lack of evidence as how decimalization is done.

Based on rearrangement, multiplication $2 \cdot 6$ represens iconically this way



$2 \cdot 6 = 12$

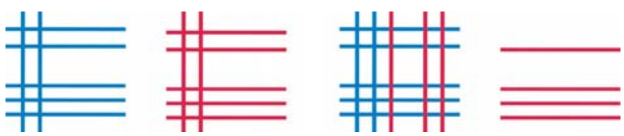
The products 4 · 6 and 6 · 6 are then represented as two and three copies of the above rearrangement, placed one below the other, respectively. The remaining case 3 · 6 represents as



3 · 6 = 18

In these, and in all following cases, the red sticks are spent for completion of blue-stick arrangements belonging to the same row of such configuarations.

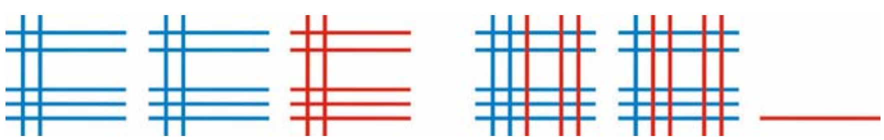
11. 5. 4. Multiplication by 7. Multiplication 2 · 7 represents as follows



2 · 7 = 14

and placing two and three copies of this representation one below the other, representations for 4 · 7 and 6 · 7 are obtained respectively.

The representation for 3 · 7 is



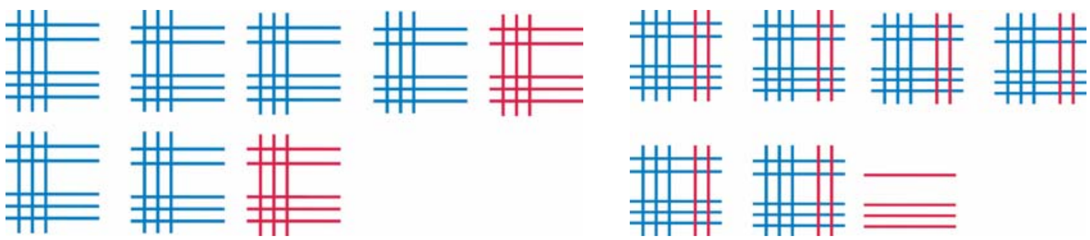
3 · 7 = 21

As an alternative representation for 6 · 7 are two copies of „3 · 7“ representation placed one below the other. Attaching to this representation an extra row with a „blue 7“, a representation for 7 · 7 is also obtained.

11. 5. 5. Multiplication by8. The multiplications 2 · 8, 3 · 8, 4 · 8 and 5 ·8 are iconically represented as eight-stick arrangements standing in one row. For example, in the case of 3 · 8, we have

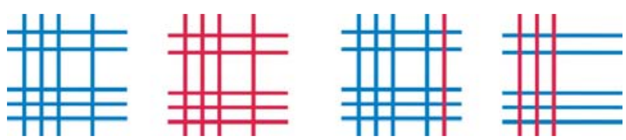
3 · 8 = 24

For 6 · 8, 7 · 8 and 8 · 8, another row representing 1 · 8, 2 · 8 and 3 · 8 is attached to the „5 · 8“ representation respectively. Thus, for example, multiplication 8 · 8 is represented as follos



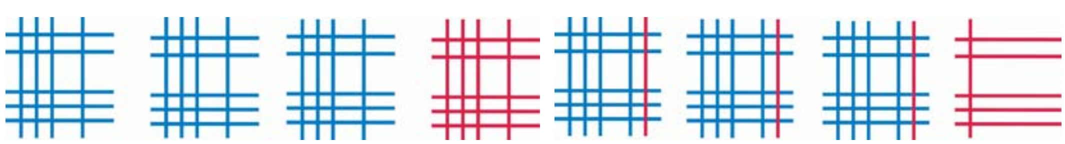
8 · 8 = 64

11. 5. 6. Multiplication by 9. Iconic representations of multiplication by 9 are all one row configurations and they go, as follows



2 · 9 = 18

. . .



4 · 9 = 36

11. 5. 7. A minor observation on multiplication cases. Looking at the given iconic representations, it is easy to select somewhat simpler cases of multiplication, which should also be learnt first and those somewhat more complex, which should be learnt after the former cases have been acquired. Thus, children should learn:

at first: $2 \cdot 5$, $4 \cdot 5$, $6 \cdot 5$, $8 \cdot 5$ and then: $3 \cdot 5$, $5 \cdot 5$, $7 \cdot 5$, $9 \cdot 5$; at first: $2 \cdot 6$, $4 \cdot 6$ and then: $3 \cdot 6$, $6 \cdot 6$; at first: $2 \cdot 7$, $4 \cdot 7$ and then: $3 \cdot 7$, $6 \cdot 7$ and, as the last, $7 \cdot 7$; at first $2 \cdot 8$, $3 \cdot 8$, $4 \cdot 8$, $5 \cdot 8$ and then: $6 \cdot 8$, $7 \cdot 8$, $8 \cdot 8$.

In order to have these iconic representations visually present for children, they should be reproduced in the bigger format (without accompanying equalities) and stuck on the walls of classroom to stay there for some time. Then, if happens (and probably will often) that a child has not memorized a case of multiplication yet, the most instructive reaction of a teacher would be to say: cast a glance and see how much it is.

11. 6. Division. In the setting of concrete didactical elaboration of division everything should also start with concrete examples , say as these two are:

(i) There are 12 apples to be shared between 4 boys. How many apples does each of them get?

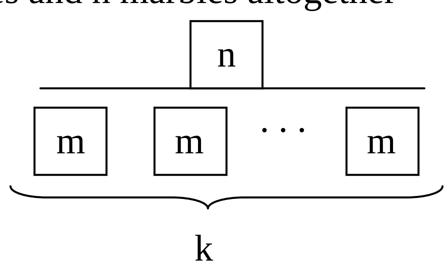
(ii) There are 12 apples. To how many boys can I give 3?

If simple equations $k \cdot x = n$ and $x \cdot m = n$ have been learnt and solved (by guessing solutions based on the already memorized multiplication table), then such a skill can be usefully employed to give the meaning to division. Working the first example, the equation $4 \cdot x = 12$ is solved and the second one, the equation $x \cdot 3 = 12$. Thus, division is nothing more than searching for an unknown factor (and can be intensively performed in implicit form as a subtopic of multiplication).

Respecting a psychological difference existing in comparison of these two examples, division is considered as quotition (from Latin, *quotes* „how many“) in the former case and as partition in the second case.

We leave aside subtle questions of didactical transposing since they can not be answered by serving recipes but directly, writing a text-book.

11. 6. 1. The same schemes. In the same way how addition and subtraction are related operations, multiplication and division are related operations as well. It also means that the meaning of these two operation is based on the same schemes and only the associated numerical data are different. For instance, having k boxes, in each m marbles and n marbles altogether



the following two kinds of problems can be assigned:

(i) Given n (total number of marbles) and k (number of boxes), find m (number of marbles in the box).

(ii) Given n (total number of marbles) and m (number of marbles in each box), find k (number of boxes).

These two kind of problems constitute the intuitive ground of division. Seaking for m in the former case, we react writing n:k and for k in the latter writing n:m. The first aspect of division (under (i)) is called *quotition* and the second (under (ii)) *partition*.

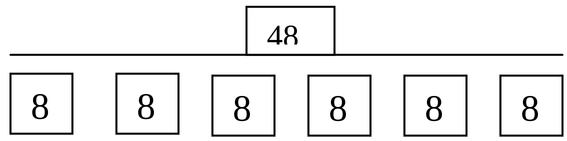
Division as a didactical topic also includes decimalization, i.e. calculation of quotients. But here, all cases of calculation are very simple and those, as for example, 72:9, 63:7, etc. are directly based on the memorization of multiplication table.

11. 6. 2. Relating two operations. A superficial approach to the early algebra views equations as a means for solving problems. In the cases when such a view prevailed, a situation arose in which, either this topic has never been on curricula of the first classes of primary school or, if existed, it has been banished from them.

On the other hand, when equations are seen as an indispensable tool for realization of the important didactical objective of development of the idea of variable, then their most simple forms suffice. Used for posing inverse problems and solved by the „method" of search for hidden (unknown) numbers, equations constitute a litlle piece of syntactic apparatus that can also be suitably used for elaboration of some didactical units of primary school mathematics. Furthermore, it is in no way necessary to prove that, exactly like grown-ups, children also enjoy to use symbols, understanding their function with ease.

An elegant way to express the meaning of the quotient n:k is to treat it as x in the equation $k \cdot x = n$. When the equation $k \cdot x = n$ is written as $x = n:k$ and vice versa, then the relationship between multiplication and division is expressed best. This, of course, does not mean that it would be done formally and in this condesed way. What it means is that, at the biginning, calculation of a quotient as, for example, 63:7 is done writing $x = 63:7$ and then, rewriting $7 \cdot x = 63$ and searching the multiplication table for such x, etc. Afterwards, when, for example, an equation as $17 \cdot x = 306$ is solved writing $x = 306:17$ and dividing, the multiplicative form of equation transforms into the divisional one. In each case, this is an activity of relating these two forms of equations with motivation.

Based on meaning, the relationship between multiplication and division elaborates procedurally – doing a number of appropriate examples. For instance, this scheme with all numerical data is present:



Reading text, children fill in the correct numbers:

In each of 6 boxes, 8 marbles. It is __ · __ = ___ marbles.

In 6 boxes, 48 marbles. In one box ___:__ = __ marbles.

In each box 8 marbles and 48 marbles altogether. The number of boxes is ___:__ = __.

When you have found that 6 · 8 = 48, then without calculating, you can write 48:6 = __ and 48:8 = __.

When you have found that 48:6 = 8 (or 48:8 = 6), then without calculating, you can write 8 · 6 = ___ (or 6 · 8 = ___) and 48:8 = __ (or 48:6 = __), etc.

The characteristic aspect of this kind of examples is that all three numbers are given and the emphesis is not on calculation but the ways how these numbers are related by means of these two operations.

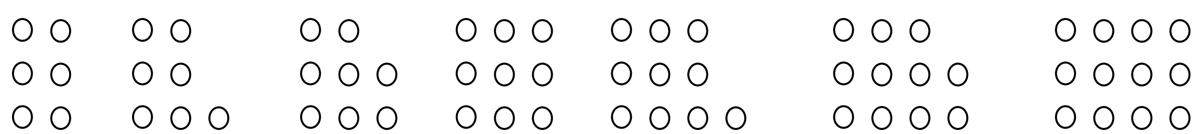
Expressing exactly how multiplication and division are interrelated, we say that, whenever one of the three relations

$$k \cdot m = n, \quad n:k = m, \quad n:m = k$$

is true , the other two are true as well.

11. 6. 3. Divisibility by 2, 3, ... ,9. Independently of its possible place in a specific curriculum, divisibility by 2, 3, ... ,9 can also be considered as a reasonable introduction to the division with remainder as a somewhat more complex topic of the block 1 - 100. For that reason, this topic deserves to be elaborated more intensively. We will only treat here the case of divisibility by 3 in more detail (and the remaining cases can be treated analogously).

Comprehending additively multiplicative structure of the series of the folloing schemes



3 · 2 3 · 2 + 1 3 · 2 + 2 3 · 3 3 · 3 + 1 3 · 3 + 2 3 · 4

the expressions representing the numbers of o’s in each of them have been written.

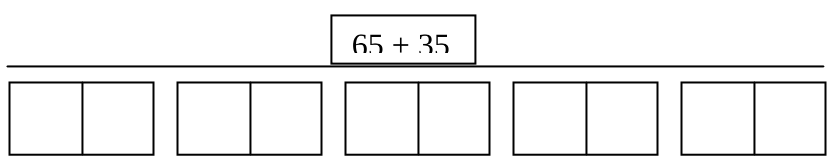
First, children calculate $3 \cdot 2 = 6$, $3 \cdot 2 + 1 = 7$, $3 \cdot 2 + 2 = 8$, $3 \cdot 3 = 9$, $3 \cdot 3 + 1 = 10$, $3 \cdot 3 + 2 = 11$, $3 \cdot 4 = 12$. Then, they find that 6, 9 and 12, when divided by 3 give 2, 3 and 4 respectively. They (together with the teacher) also find that the numbers 7, 8, 10 and 11can not be divided by 3, because 1 remains, 2 remain, ...Proceeding, the teacher announces that the numbers which can be divided by 3 are called *divisible* by 3 and writing

$$3 \cdot 2 = 6,\ \ 3 \cdot 3 = 9,\ \ 3 \cdot 4 = 12,$$

he/she requires of children to continue that sequence of equalities. Visual stimulation of seeing such numbers represented by schemes having columns of threes is certainly helpful (and by a suitable teacher’s story easily made representing something real). Equally good is the counting in threes begining at 3, 30, 60 or 90.

In this and remaining cases (which are treated similarly) divisibility is more an activity of actual dividing and not derivation of well-known rules expressing divisibility in terms of digits of a number.

11. 6. 4. Rules of division of sums and differences. A teacher (together with his/her class) establises these rules doing a number of examples chosen for that purpose. Everything starts with description of a scheme: There is a row of 5 equal, two-box packages containing blue and red marbles arranged in the boxes separately. The total number of blue marbles is 65 and of those red 35.



The number of marbles in one package is $(65 + 35):5$, (or place holders are used (___ + ___):__, when the enforcement of answers is planed). The number of blue marbles in one box is $65:5$ and of those red $35:5$. Altogether, it is $65:5 + 35:5$ marbles. Without calculating, the equality

$$(65 + 35):5 = 65:5 + 35:5$$

is obtained.

Doing such and similar examples, the *rule of divison of sums* is established. Then, it can be used procedurally and be formulated rhetorically.

In general, there will be a row of k equal, two-box packages containing together n_1 blue marbles and n_2 red ones. Then, proceeding in the same way as above, the symbolic form of the rule

$$(n_1 + n_2):k = n_1:k + n_2:k$$

is obtained, based again on meaning and not on generalization upon specific cases.

Without going into analogous details and taking the same scheme but changing data: n is total number of marbles, n_2 the number of those red and then, searching for the total number of blue marbles, the symbolic form of the *rule of division of differences*

$$(n - n_2):k = n:k - n_2:k$$

is obtained.

Needless to say that these rules hold true only when summands in the former case and both, minuend and subtrahend in the latter case are divisible by k, and that is what always has to be stated including the rhetoric forms of these rules.

The most important, and in the same time easiest cases of calculation are those which are nothing more but entries from the multiplication table written as quotients (as, for instance, 72:8, 56:7, etc.). Besides them, the cases of division by one-digit divisors have a place in this content. For example, divisions as

$$72:3 = (60 + 12):3 = 60:3 + 12:3 = 20 + 4 = 24,$$
$$91:7 = (70 + 21):7 = 70:7 + 21:7 = 10 + 3 = 13, \text{ etc.}$$

are done, where application of the rule of division of sums works with motivation. Needless to say, such examples should always be given in the programmed form.

11. 6. 5. Division with remainder. Considered as a purely mental performance and based on the long-term memorization of the multiplication table, division with remainder is bounded to the following cases:

divisor	set of dividends
2	1, 2, ... , 19
3	1, 2, ... , 29
. . .	
9	1, 2, ... , 89

For the sake of illustration, let us consider the case of division by 4. Each of the numbers 1, 2, ... ,39 is either divisible by 4 or there exists a unique greatest multiple of 4, smaller than that number. Also helped with a visual representation

○ ○ ○ ○ ○
○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○

$$4 \cdot 5 + 2$$

in the case of number 22, the greatest multiple of 4 smaller than 22 is $4 \cdot 5$. Then, $22 = 4 \cdot 5 + 2$ and this equation is read: 22 divided by 4 is 5 and 2 remain, what is written as a shorthand

$$2\underline{2}:4 = 5$$
$$\;2$$

To develop necessary skill, a number of exercises should be done:

Find the greatest multiples of 4 smaller than the given numbers and then, complete the given equalities:

(i) For 7, the greatest multiple is $4 \cdot$ __ and $7 = 4 \cdot 1 +$ __,

(ii) For 38, the greatest multiple is $4 \cdot$ __ and $38 = 4 \cdot 9 +$ __, etc.

(iii) Read the equality $7 = 4 \cdot 1 + 3$ and write it shortly .

(iv) Read the equality $38 = 4 \cdot 9 + 2$ and write it schortly.

etc.

(This is an example of the use of place holders where requirements and correct responses are combined).

After such and similar examples have been done, oral exercises of division with remainder should be included as well.

Some unclear shorthand forms, including mathematically nonexisting expressions as, for example, 38:4 = 9 r.2, 38:4 = 9,(2), etc. are often seen in text-books. However the shorthand form given here is a notation representing procedure of long division in the simplest case (and not equality of expressions).

11. 7. Calculation cases reconsidered. In the susectin 10. 3 of this paper, the following two narrative rules have been formulated:

1. Units are added to units and tens to tens.
2. Units are subtracted from units and tens from tens.

Using colours and as a technical device, digits of numbers denoting units will be in red and those denoting tens will be in blue.

Following the above rules, in the first case of calculation, for instance, we have

$$38 + 5\ 6 = 80 + 14,$$

and in the second

$$72 - 49 = 60 - 40 + 12 - 9.$$

To get decimal denotation of 94, we still have to add tens in the former case and, making the rule of subtraction feasible, first we had to transform 72 into 60 + 12 in the later case. To condese such procedure of carrying or borrowing one ten, we design a two-place holder and write the sums, as follows

$$80 + 14 = \overset{1}{\boxed{8\,|\,4}}\,,\quad 60 + 12 = \overset{1}{\boxed{6\,|\,2}}$$

Then, we can proceed calculating this way

$$38 + 56 = \overset{1}{\boxed{8\,|\,4}} = 94,\quad 72 - 49 = \overset{1}{\boxed{6\,|\,2}} - 49 = 23,$$

so respecting litterally the above rules.

To make this little piece of technique run easily , exercises like these:

$$50 + 11 = \boxed{\,|\,}\,,\quad 70 + 8 = \boxed{\,|\,}\,,\quad 80 + 19 = \boxed{\,|\,}\ \text{etc.}$$

$$\overset{1}{\boxed{5\,|\,1}} = \_\_\_ + \_\_,\quad \overset{1}{\boxed{7\,|\,8}} = \_\_\_ + \_\_,\quad \overset{1}{\boxed{7\,|\,8}} = \_\_\_ + \_\_,\ \text{etc.}$$

should be done first. Then, children should be doing examples as:

$$33 + 57 = \boxed{\,|\,} = \_\_\_,\quad 62 + 19 = \boxed{\,|\,} = \_\_\_,\ \text{etc.}$$

$$91 - 58 = \boxed{\,|\,} - 58 = \_\_\_,\quad 64 - 39 = \boxed{\,|\,} - 39 = \_\_\_,\ \text{etc.}$$

without too many hints: „borrow“, „carry“.

This technique will be used more extensively in the next section, where its greater effects will be seen.

At the end, let us say the block of numbers 1 – 100 is now an enriched structure, that is symbolically expressed by writing

$$\{N_{100}, +, -, \cdot, :, =, <\}.$$