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A VERSATILE TOOL TO PROMOTE LINK BETWEEN CREATIVE PRODUCTION AND CONCEPTUAL UNDERSTANDING

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Abstract. With respect to constructivist views on the nature of mathematical knowledge and the genesis of heuristic processes in the mind of the learner, the abundance of problems, richness of ideas and students' possibilities and power to develop intuition have to be redefined by utilizing versatile technological tools. This paper highlights how link between creative production and conceptual understanding may be promoted by use of a progressive pocket computer. It focuses on learning based upon the interplay of different representations of mathematical objects, the use of which would improve problem solving abilities as well as the understanding of underlying mathematical concepts. We describe this kind of learning and examine its empirical values by using a modification of a classical extreme value problem.

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Introduction

Knowing which thinking tools (heuristics) have been especially productive for the development of mathematics over a long range of time and in different cultures is an important issue for mathematics education and its management. When shifting the focus to mathematics teaching, we should know how mathematical knowledge and mathematical thinking might come into student's heads, and into their lives and actions. If we carry out a study of the history of mathematics as a long-term study of cognitive and motivational processes and activities resulting in new mathematics, we may identify eight main motives and activities yielding new mathematical results at different times and in different cultures for more than 5000 years [15].

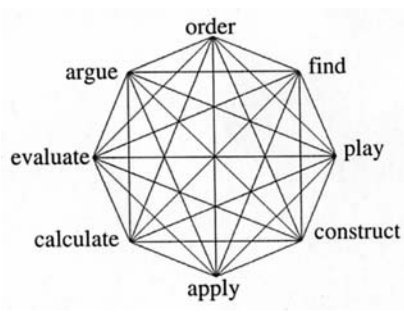


Fig. 1. Activities and thinking tools particularly successful in producing new mathematics [15, p. 42]

These activities presented in Figure 1 may be taken as an element of an educational framework for contemporary teaching and learning of mathematics, which may not be subjected too strongly to recent modern waves and rapid changes of fashions known under labels “New Math”, “Back to Basics”, and “Applications”. With respect to Figure 1, the “find”-corner represents heuristic activities (cf. [12]). Among them is the “Changing representation” activity that this contribution focuses on. As this activity not only empowers problem solving but also promotes concept formation and understanding, we should have now a closer look on procedural and conceptual learning and knowledge.

Interplay of conceptual and procedural knowledge

Procedural (mathematical) knowledge often calls for automated and unconscious steps, whereas conceptual knowledge typically requires conscious thinking. We adopt the following characterization of [4, p. 141]:

- *Procedural knowledge* ($\boldsymbol{P}$) denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation forms. This usually requires not only the knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.
- *Conceptual knowledge* ($\boldsymbol{C}$) denotes knowledge of and a skilful “drive” along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

The history of mathematics supports many educators’ view that $\boldsymbol{P}$ enables $\boldsymbol{C}$ development. The logical background is *genetic view* ($\boldsymbol{P}$ is a necessary but not sufficient condition for $\boldsymbol{C}$) or *simultaneous activation view* ($\boldsymbol{P}$ is a necessary and sufficient condition for $\boldsymbol{C}$). An instructional interpretation is: Utilize $\boldsymbol{P}$ and reflect on the outcome. Haapasalo & Kadijevich [4, pp. 147–150] call this position *developmental approach* as it reflects the phylogenesis of mathematical knowledge as well as its ontogenesis. Even Figure 1 gives us reason to assume that the making of mathematics has been primarily guided by pragmatic aspects. Procedures were devised first; conceptual clarifications have been undertaken latter - or are still waiting to be done [15]. Very often needs for concept-formation and their clarification emerged within the process of problem solving[1].

In the individual development of mathematical knowledge it seems that again $\boldsymbol{P}$ develops faster than $\boldsymbol{C}$. Students’s behaviour very often reveals the existence of powerful implicit concepts and theorems that may be called ’concepts-in-action’ and ’theorems-in-action’. Such knowledge cannot be properly called conceptual, even though ’objects-in-actions’ may eventually be conceptualized.

Most educators assume the dependence of $\boldsymbol{P}$ on $\boldsymbol{C}$, however. Their assumption may be briefly summarized as: it is $\boldsymbol{C}$ that enables $\boldsymbol{P}$. An instructional implication

[1] See, for example, the emergence of various versions of the concept of ’polyhedron’ [9]

is: Build meaning for $\boldsymbol{P}$ before mastering it. Haapasalo & Kadijevich [4, pp. 150–154] call this position *educational approach* since it seems to fulfil educational needs, typically requiring a large body of knowledge to be understood and transferred. The educational approach may be supported by *dynamic interaction view* ($\boldsymbol{C}$ is a necessary but not sufficient condition for $\boldsymbol{P}$), or the above-mentioned simultaneous activation view. It is the presence of metacognition that is crucial to $\boldsymbol{C}$ development and for this reason $\boldsymbol{P}$ acquisition is generally more accessible to human beings than $\boldsymbol{C}$.

A sophisticated interplay of these two approaches can be illustrated in a flourishing way when the applied framework of knowing and learning is linked to the dynamic interaction and simultaneous activation views (see [2][2]). By acknowledging these views, this contribution presents main ideas of promoting link between creative production and conceptual understanding by means of the change of mathematical representation, this activity being utilized within a progressive pocket computer. These ideas are emphasized for a modification of a classical extreme value problem involving quadratic functions (parabolas). In problem solving, in general, change of representation (e.g., from symbolic to graphic, and vice versa) may be useful to improve the problem solving process as well as the understanding of the concepts under consideration. In our example, constructing conceptual knowledge around "parabola" profits from utilizing different representations whose utilization are useful in solving corresponding problems. So we can say that 'simultaneous activation' and 'changing representation' may be seen as two aspects of the same educational enterprise.

Overview of ClassPad operations

For more than 25 years, mathematical objects and concepts can be represented on personal computers by symbolic expressions and graphs. However, students need a lot of conceptual and procedural knowledge for being able to do this. This aspect becomes very important when mathematical software becomes more and more versatile. Our educational problem today is that before students apply highly sophisticated computer tools, their informal thinking should be fostered through stimulating problems and knowledge/skills profitable learning environments respecting demands of modern constructivist theories on learning. In such situations modern technological tools might reinforce and enlarge the scope of creative activities. We illustrate such activities by utilizing *ClassPad 300*, a modern pocket computer made by *Casio* (see `http://www.classpad.org`; [5]).

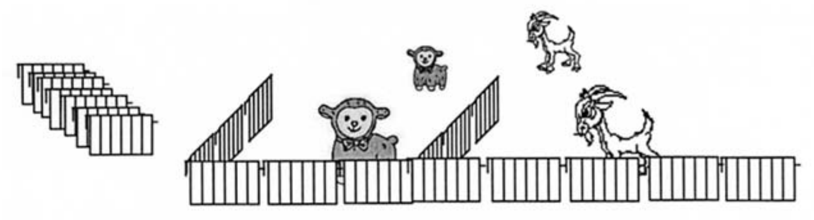
Most *ClassPad* applications support simultaneous display of two windows: *Main Application* work area and *Currently Application* area (*Geometry*, *Statistics* or *Spreadseets*, for example), allowing drag and drop activities (i.e. copy and paste actions) and other operations between the two windows.

[2]A learning software can be freely downloaded from `http://www.joensuu.fi/lenni/programs.html`.

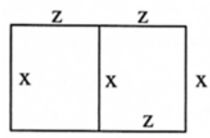
Empowering heuristic thinking

Let us consider the following worksheet for upper-secondary (e.g., ninth-grade) students[3] including an extreme value problem:

A farmer has 200 fence-elements, which he can carry and put together into two connecting corrals for his sheep and goats. After the animals will have eaten all grass, the corral will be rearranged.



The form of the corral should be a rectangle (see figure at the right). Each element of the fence has a breadth of one meter. There are many possibilities for the farmer to construct a corral: to use more "thin" or "thick" rectangles. In which way he should arrange the fence-parts for the corral to get maximal pasture for the sheep and the goats?



1. Some hints for your work (cf. [12]):
   - consider special cases at first,
   - construct and fill out a table in a systematic way,
   - put the values from the table into a diagram,
   - set up and check conjectures about the best fitting shape of a rectangle,
   - use expressions with variables.
2. Think now, that you have no longer fence-elements of one meter but a "continuous" fence material. Can you find now a solution with even larger pasture for the animals?

When working in this problem field there might be stimulated a broad variety of thinking processes as given below.

At the beginning, students can approach the problem by some simple model, e.g., matchboxes as fence-elements. They can start with a small number of matchboxes, divisible by four, and look for the best rectangle without separation. By doing this, students can generate conjectures (as square as best from) as well as reasonable trial and error methods. Then, they can go on to the "separated" rectangle and work in small groups with a worksheet. Figure 2 shows different

[3]Of course it might be used as an example in student teacher education, as well. Cf. the paper of Rehlich [13].

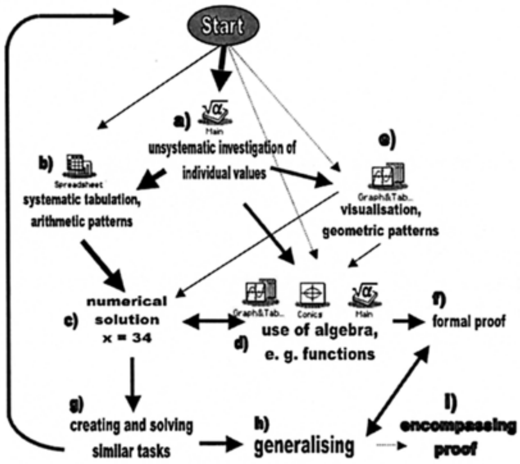


Fig. 2. Overview on heuristics utilizing *ClassPad*

possibilities of students' activities and some of their potential sequences. The small icons refer to respective adequate menu of a *ClassPad*. The dashed or thin lines represent small, thick lines high transition-probability.

When approaching the problem in many different ways students discover mathematical patterns to understand their relations. After sufficient exploration, the pursued approaches and obtained results are collected to trigger further iterations and representations. We present shortly some heuristics, and show how the *ClassPad* properties allow versatile interplay between arithmetic, algebra and geometry (the letters from a to i refer to Figure 2).

(a) Trial and error

The first approach might be to make a special hypothesis (e.g., the shape of the fence might be a square) and to test it (cf. footnote 4).

(b) Systematic tabulation

By making a table students can get an interesting insight: If you take some values for x, you get some values for the breadth $2z$ and for the area of the corral. The sequences of differences reveal interesting arithmetical patterns, so that you can calculate all consecutive values very easy in both directions (see Table 1).

Table 1. Utilizing systematic tabulation by paper and pencil method

x	16	18	20	22	24	26	28	30	32	34	36
2·z		73	70	67	64	61	58	55			

-3 -3 -3 -3 -3 1st sequence of differences is constant

area		1314	1400	1474	1536	1586	1624	1650			

86 74 62 50 38 26

-12 -12 -12 -12 -12 2nd sequence of differences is constant

decreasing growth

However, *ClassPad* offers a much more easier method: just use spreadsheets. Figure 3a shows a trial method by x in column A. Columns B and C represent z and the area, respectively[4]. Figures 3b and 3c illustrate how easily student can move to graphic representation without understanding the links to algebra.

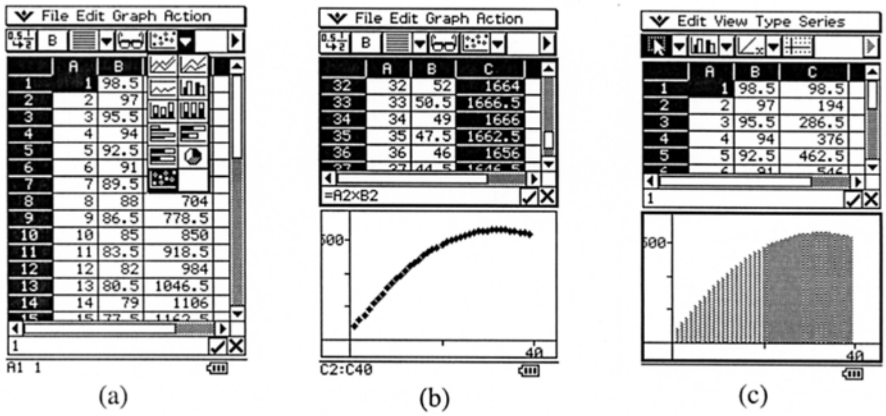


Fig. 3. Utilizing spreadsheets with graphic representation options

(c) Numerical solution

Through getting an insight of arithmetic structures, the table may lead us to the solution $x = 34$.

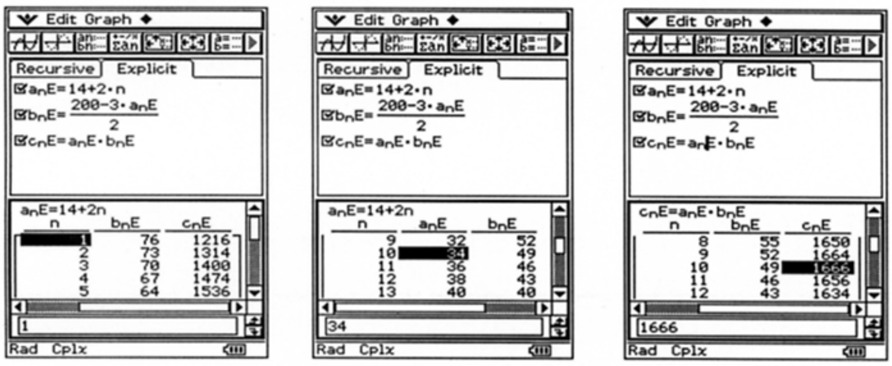


Fig. 4. Utilizing explicit recursive method with *ClassPad*

(d) Using algebra

By using explicit expressions for the depth x $(= a_n)$, the breadth $(= b_n)$ and the area c_n, students can immediately get a domain where the maximum value might

[4]There are many opportunities for students to notice that the number of fence elements x must be even. Furthermore, they can optimize the whole area ore one half of it (as done above).

be found (see Figure 4). Of course, they have to think what kind of input they have to type for the explicit expressions of the corresponding magnitudes. However, after some “informal” experiences as described above, it should not be too difficult to experience the transformation from symbolic into “numerical” representation (sequence of numbers in a table).

(e) Making diagrams by using simultaneous activation

The students can take the values from table 1 (without computer) to draw a graph by using conventional plotting method. When using *ClassPad*, it is much easier to move from the expressions of the magnitudes to graph plotting. The above-mentioned investigations can be done with much more flexibility by utilizing a simultaneous activation of symbolic, table-bounded and graphic representations. Students can utilize the versatility in choosing different kinds of *ClassPad* operations. By looking at Figures 3 and 5 some special values might help to unravel the shape of a parabola. Students might become aware of the functional aspect and they might use their prerequisite knowledge. Furthermore, the graph might trigger some other conjectures.

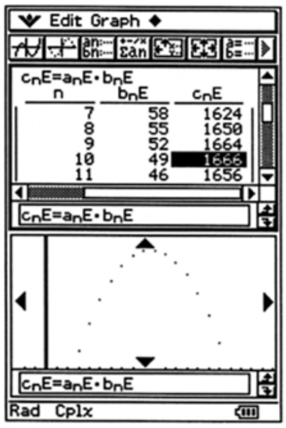


Fig. 5. Moving from algebra to graphic representation

Toward more sophisticated mathematical models

If the students become aware that the problem can be modelled by a quadratic function, they may apply additional prerequisite knowledge with paper and pencil method or by using computer. They can calculate the area $A(x) = x(200 - 3x)/2$ and recognize this as an equation of a parabola. Because of the symmetry property of the parabola, they can conclude that the vertex is located in the middle of the x-axis intersections 0 and 200/3, so that the x-coordinate of the vertex is 100/3. Then they might take the closest integer $x = 34$ and check it. Of course, they can also make an algebraic transformation of this equation, which can easily yield the coordinates of the top of the parabola. In this way the students get a solution including a *formal proof* (cf. in Figure 2; the letters g–i below refer to the same figure).

The insights and methods gained by the above mentioned activities might stimulate the creation of similar problems (g). When working on these problems students might discover that—independent from local circumstances—all solutions of analogical problems can be characterized by the condition “*use the same sum of border length vertically as found horizontally*” (h). Figure 6 shows a representative *encompassing proof* of this necessary condition (i). This leads easily to universal valid method to improve constellations without this characteristic by the dual problem. We have not found in the literature this kind of generalization method for our starting problem, even though solution methods of familiar problems have been applied throughout the history of mathematics.

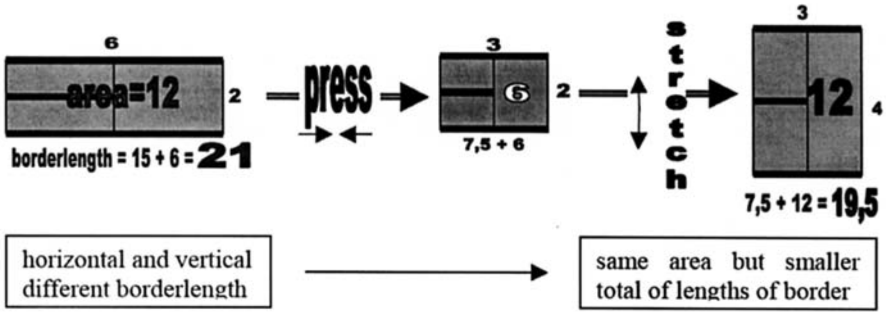


Fig. 6. Moving to generalization with encompassing proof

Closing remarks

Mathematics teaching in school is mainly concentrated on the drilling of procedural routines, whereas mathematics at university level is mainly based on conceptual approach with quite poor procedural implications. Both of these polarizations utilize scientific language and handling of old, clearly formulated and unambiguous questions. It is our position that one of the biggest challenges for mathematics education is to relate conceptual and procedural knowledge. To achieve this end, an analysis of the history of human thinking processes during more than 5000 years might be useful. Our example show that all eight components highlighted in Figure 1 can become a vital part of mathematical learning processes. They should be taken seriously (in a balanced way, of course) no matter which professionals our students are going to be: mathematicians, engineers, teachers or others. Concerning teacher education, we probably would not get any shifting from the traditional teaching culture to a progressive one utilizing dynamic problem-solving environments, if same kind of processes, including new technological solutions would not be a vital part of teachers' own studies (cf. [10, p. 43]).

Kadijevich [8] points out four areas, which have been neglected in research on mathematics education: (1) promoting the human face of mathematics; (2) relating procedural and conceptual mathematical knowledge; (3) utilizing mathematical modelling in a humanistic, technologically-supported way; and (4) promoting technology-based learning through applications and modelling, multimedia design, and on-line collaboration. His approach offers special challenges to utilize modern technology in all its forms. However, its sustainability should be also evaluated as done by Haapasalo and Siekkinen [6]. They underline that dynamic problem-based learning modules offer new features and challenges not only for teaching and learning of mathematics but also metacognitive skills. But, as we have seen from the studies of Pesonen et al. [10], new technology can cause cognitive problems for students, and the use of interactive modules would not bring special advantages in teaching without an appropriate pedagogical framework and reflective tutoring. Other problems concerning interactive graphic representations, for example, are reported by Sierpinska et al. [14]. They found that students showed quite surprising ways in interpreting dynamic figures, which seemed to differ from the classical paper-and-pencil representations. Furthermore, moving from the old studying culture towards a modern technology-based one can cause not only cognitive but also emotional and social problems. Referring to the findings of Jarvela & Haapasalo [7], it is appropriate to assume that interactive learning environments should be tailored to fit students learning profiles. Whilst conceptual learners might profit from exact definitions, procedural learners would profit from their more or less procedural thinking. The most problematic group could be the group of procedure-bounded learners, who probably would need special arrangements to get free from their spontaneous naive procedural thinking. Thus, the question about 'minimalist instructions' to prevent the conflict between conceptual and procedural knowledge concerning as well mathematics as the use of technology (cf. [3]) becomes crucial, especially in the case of *ClassPad* type versatile tools, whose menus contain a huge amount of conceptual knowledge.

Perhaps the most promising aspect of technology-based learning is to utilize the principle of simultaneous activation of conceptual and procedural knowledge (cf. [2], and [5]). This allows the teacher to be freed from the worry about the order in which student's mental models develop when interpreting, transforming and modelling mathematical objects. Our examples hopefully show that more or less systematic pedagogical models connected to an appropriate use of technology can help the teacher to achieve this goal by allowing free architecture of learning. The following quotation of the famous physicist E. Mach "*You cannot understand a theory unless you know how it was discovered*" (see [1]) might help to highlight the importance of linking the process of discovery (cf. Figure 1), understanding and conceptualization incorporated in theory building. *ClassPad* type technology can be used for increasing new kind of complexity for the mathematical content—being an essential element when building a bridge between school and university and when scaffolding mathematics making both inside as outside educational institutions.

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