THE TEACHING OF MATHEMATICS
2004, Vol. VII, 2, pp. 71-93

A BROADER WAY THROUGH THEMAS OF ELEMENTARY SCHOOL MATHEMATICS, VII

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Abstract. An extension of the block of numbers 1 – 100 to the block of numbers 1 – 1000 is considered. The reason for singling out this block is a thorough elaboration of the procedures of decimalization. Represented as sums of hundreds and one- or two-digit numbers, addition and subtraction of three-digit numbers is reduced to easy and already known cases of performing these operations. Represented as sums of units, tens and hundreds, these two operations are also carried out on such summands. To the latter case, the shorthand forms of performing these operations on digits, are attached. Bound to the cases of one-digit multipliers or divisors, multiplication and division are treated similarly.

By analogy, these acquired skills and understanding are expected to be carried over in cases of all other numbers, when operations are chiefly performed on digits.

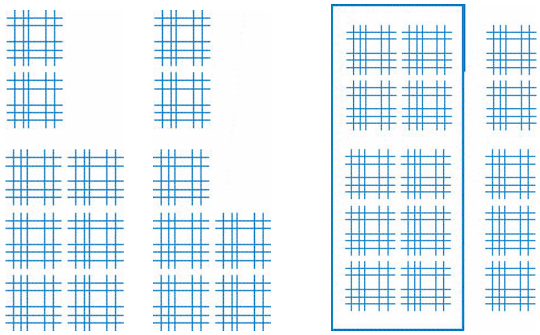
At the end, we summerize main points of our approach to the elaboration of arithmetic, being exposed in all seven patrs of this paper that run under the same title.

ZDM Subject Classification: F 32; *AMS Subject Classification*: 00 A 35.
Key words and phrases: Block of numbers 1 – 1000, Decimalization.

12. Block of numbers 1 – 1000

The number blocks that we have already considered are the structural entities that are singled out for achieving specific didactical tasks. Particularly important are structural features of blocks 1 – 20 and 1 – 100, within whose ranges the addition and multiplication tables are built respectively. The block 1 – 1000 that we will treat in this section is singled out with the motivation to make possible further development of the skill in decimalization that children acquire by practicing it on longer (three digit) decimal notations. And as it will be seen, the way of extension and the additive structure of this block closely resemble those in the case of the block 1 – 100. Namely, in the first step the extension goes up to 200 and then, it covers the whole range of numbers of this block. Among didactical tasks that are forced structurally, the multiplication by one-digit multiplier and division by one-digit divisor are also included in the frame of this block.

12. 1. **Gradually, first up to 200.** In extending the block 1 – 100, the first step consists of using the sums: 100 + 10, 100 + 20, … , 100 + 100 and the shortened decimal way of their writing: 110, 120, … , 200. The adding and subtracting of these tens is completely analogous to the way how these operations are carried out in the case of units, in the block 1 – 20. This is also the right place to start using a standard iconic sign that will represent one hundred. Such a sign that we will use is seen in the following picture



80 + 70 = 100 + 50 = 150

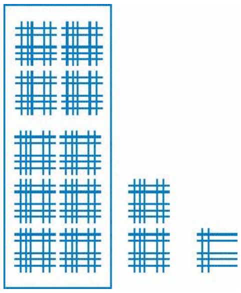
In the next step, the sums as: 100 + 4, 100 + 16, 100 + 21, 100 + 49, etc. are written shortly, using their decimal notations: 104, 116, 121, 149, etc. Exercises in writing and saying these numbers should include equating of such sums with their decimal notations and vice versa. For instance,

100 + 1 = ___, 100 + 10 = ___, 100 + 53 = ___, etc.

but also

104 = 100 + _, 140 = ___ + 40, 176 = 100 + __, etc.

These exercises can also be considered as the simplest cases of addition, when a summand is 100. Almost equally simple are the cases

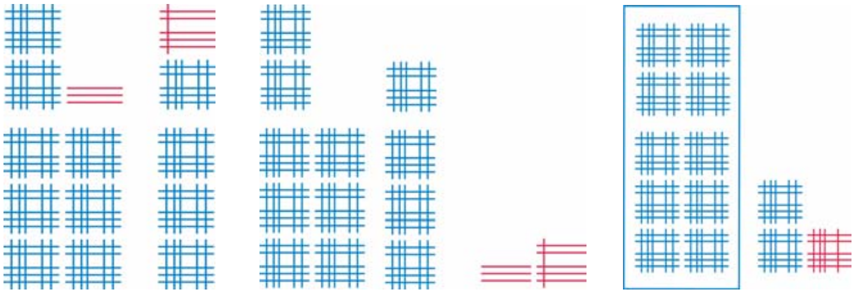


120 + 7 = 127, 150 + 6 = ___, 170 + 9 = ___, etc.

In general, the addition taken as the transformation of sums into their decimal notations is subjected to the procedural rule: units are added to units and tens to tens. To let children pick out this rule easier, when a sum is assigned the digits denoting units are coloured in red and those denoting tens in blue.

For instance, the following two examples are representative of the cases of addition without carrying one ten.

a)

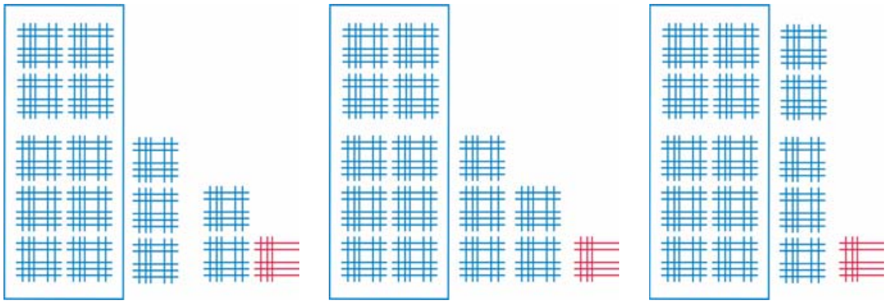


83 + 46 = (80 + 40) + (3 +6) = 120 + 9 = 129

Using the place holders, the shorthand form of this addition goes as follows

$83 + 46 = \boxed{1 \mid 2 \mid 9}$

b)



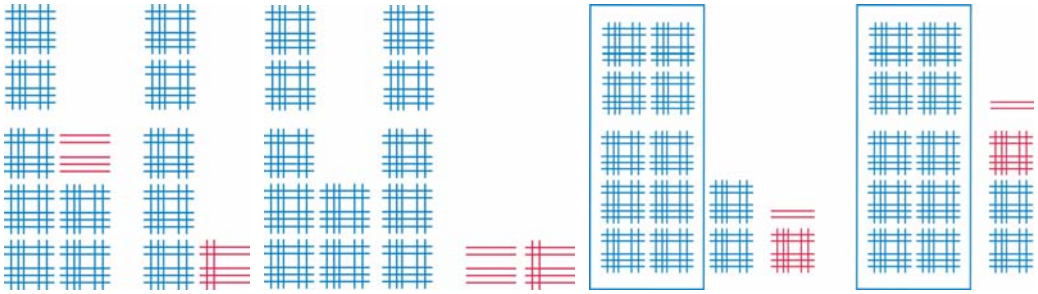
130 + 28 = (130 +20) + 8 = 150 + 8 = 158

and

$130 + 28 = \boxed{1 \mid 5 \mid 8}$

The cases of addition with carrying one ten are illustrated with the following two examples.

c)



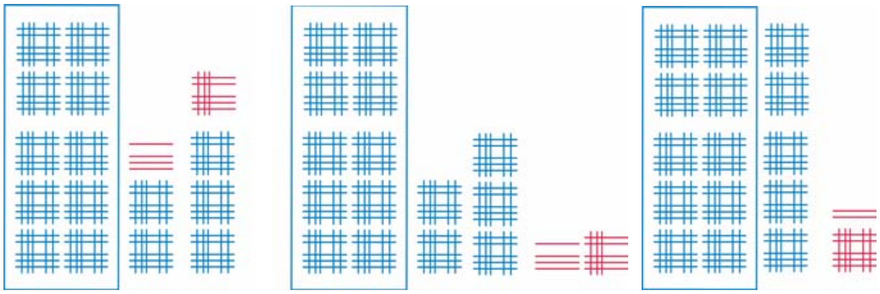
75 + 57 = (70 + 50) + (5 + 7) = 120 + 12 = 132

(Not to cause a possible confusion in correspondence between colours of sticks and those of digits, the arrangement of blue and red tens is denoted using black digits, as it is the case with 132 in this example. An exception is red zero denoting non-existence of elements.)

The shorthand calculation goes as follows

$$75 + 57 = \boxed{1}\overset{1}{\boxed{2}}\boxed{2} = 132$$

d)



$$124 \quad + \quad 38 \quad = \quad (120 + 30) \quad + (4 + 8) \quad = \quad 150 \quad + \quad 12 \quad = 162$$

and

$$124 + 38 = \boxed{1}\overset{1}{\boxed{5}}\boxed{2} = 162$$

In a number of initial examples, the shorthand addition should come at the end of the procedure of decimalization as a stenographic way of writing it down. When numbers denoted by red digits are added, the teacher should comment: “Now we add units” and he/she should also turn the children’s attention to the corresponding manipulation with the arrangement of sticks. Of course, he/she should act similarly in the case of blue digits and the addition of tens.

Subtraction usually starts together with addition, coupling the corresponding cases. After finding that

$$75 + 57 = 132,$$

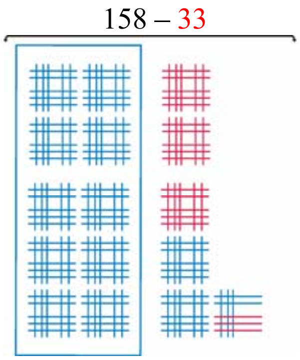
and by relating two operations, children easily do such bits of work as to complete equalities

$$132 - 75 = \_\_, \qquad 132 - 57 = \_\_,$$

but it does not help them to find these differences when they are given independently of the foregoing addition.

By suggesting here the way of elaboration of subtraction, we will use again the two-color sticks for the purpose of iconic representation. A whole arrangement will stand for a minuend, its part consisting of red sticks for a subtrahend and the set of blue sticks will represent the difference. The former two of these arrangements are standard so that their shapes clearly project the numbers that they stand for. Now, typical examples for each specific group of differences will be given.

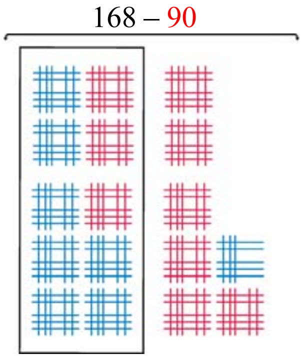
e)



$158 - 33 = \boxed{1}\boxed{2}\boxed{5}$

158 – 33 = (150 – 30) + (8 – 3) = 120 + 5 = 125

f)

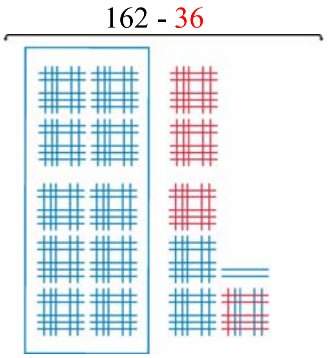


$168 - 90 = \boxed{}\boxed{}\boxed{}$

168 – 90 = (160 – 90) + 8 = 70 + 8 = 78

In both of these cases the rule: Units are subtracted from units and tens from tens, applies directly. In the examples that follow, the borrowing of one ten precedes its application.

g)



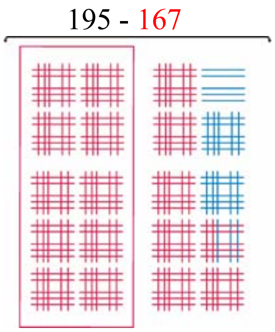
162 – 36 = (150 – 30) + (12 – 6) = 120 + 6 = 126

The teacher should accompany the shorthand calculation with his/her comment:

$162 - 36 = \boxed{1}\boxed{\overset{1}{5}}\boxed{2} - 36 = 126$

As you see in the picture, 6 units subtracted from 12 units are 6 units and 3 tens subtracted from 15 tens (one ten has been borrowed) are 12 tens.

h)



$195 - 167 = (100 - 100) + (95 - 67) = 28$

$195 - 167 = \boxed{1 \mid \overset{1}{8} \mid 5} - 167 = 28$

Iconic representations in the form of arrangements of sticks are the intuitive ground for the acceptance of the above rules. The effects of these rules are written down in the form of arithmetic expressions which are then transformed into the unique decimal notation of the sought number. Operating on digits of decimal notations, the shorthand calculations have to be related to the transformations of arithmetic expressions. But these calculations have to be practiced independently of any interpretation for the purpose of being performed automatically. In the beginning of this process, calculations should be described in words, first by teacher and then, also by children. For instance,

$84 + 47 = \boxed{1 \mid \overset{1}{2} \mid 1} = 131$

Comment: 7 units plus 4 units make 11 units, 8 tens plus 4 tens make 12 tens. Then, it is 13 tens and 1 unit;

$142 - 67 = \boxed{1 \mid \overset{1}{3} \mid 2} - 67 = 75$

Comment: We cannot subtract 7 from 2. We borrow 1 ten so as to have 12 units and 13 tens remain. Now, 12 units minus 7 units make 5 units and 13 tens minus 6 tens make 7 tens.

Of course, as the skill in calculation grows, such comments will be omitted.

12. 2. Extension up to 1000. Hundreds are introduced in the following way

The sum:	is written as:	and said:
100 + 100	200	two hundred

200 + 100	300	three hundred
	…	
900 + 100	1000	one thousand.

The equalities 100 + 100 = 200, 200 + 100 = 300, … , 900 + 100 = 1000 result from the equating of notations. Addition and subtraction of hundreds are very simple and they are similar to the case of these operations in the block 1 – 10.

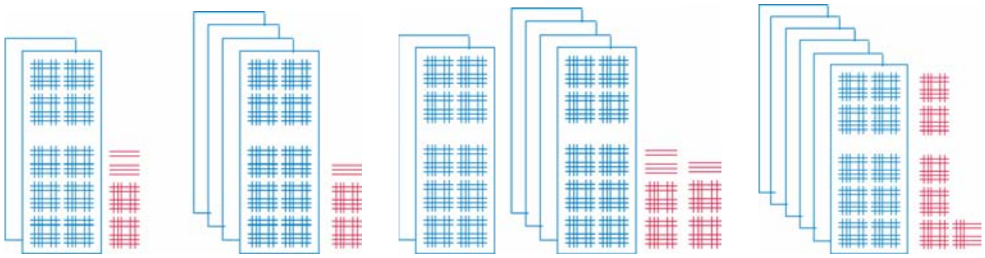
By means of a number of exercises as the following ones:

The sum:	is written as:	and said:
200 + 80	280	two hundred eighty
400 + 76	476	four hundred and seventy-six
600 + 3	603	six hundred and six, etc.

children learn to write and say all numbers up to 1000.

Addition and subtraction in the block 1 – 1000 are reduced to the performance of these operations on hundreds and on numbers up to 200. Typical cases will be illustrated by examples that follow.

a)



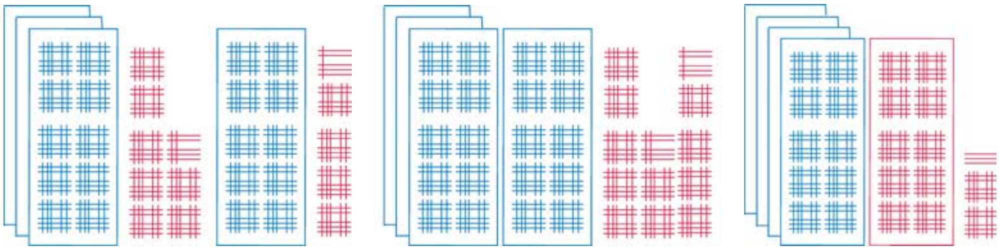
235 + 423 = (200 + 400) + (35 + 23) = 600 + 58 = 658

Based on this model, a number of exercises should be done:

345 + 212 = (___ + ___) + (__ + __) = ___+ __ = ___,

432 + 563 = …, etc.

b) This example is typical for those cases when one hundred is carried.



377 + 146 = (300 + 100) + (77 + 46) = 400 + 123 = 523.

To practice this way of calculation use, the exercises as the following ones:

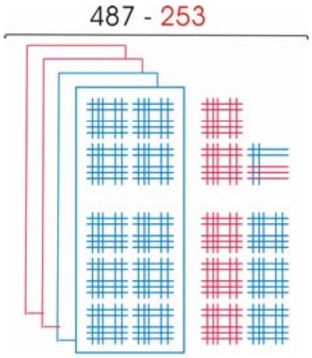
628 + 284 = (___ + ___) + (__ + __) = ___ + ___ + ___,

374 + 459 = …, etc.

have to be assigned to children.

In these exercises different colors have been used to indicate sums of hundreds (blue) and of two-digit numbers (red). In general, the use of coulors is subjected to the way how a specific didactical task is carried out and, it varies from one case to another. Thus, in the case of subtraction, the red digits are used to denote subtrahends and the blue ones to denote differences, what is then in accordance with the colour of the corresponding arrangements of sticks.

c)



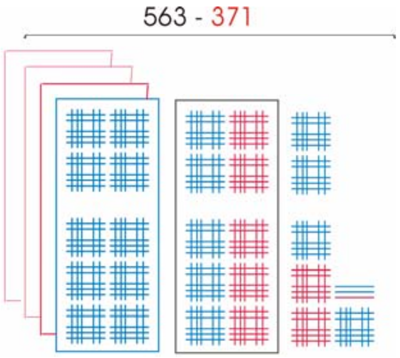
487 – 253 = (400 – 200) + (87 – 53) = 200 + 34 = 234.

This model of subtraction without borrowing one hundred, should be followed by a number of examples such as these:

683 – 251 = (600 – 200) + (83 – 51) = ___ + __ = ___,

784 – 528 = (___ - ___) + (__ - __) = ___ + __ = ___, etc.

d) The following example is a model of subtraction when one hundred is borrowed.



563 – 371 = (400 – 300) + (163 – 71) = 100 + 92 = 192.

Examples to follow are these:

861 – 495 = (700 - ___) + (161 – 95) = ___ + __ = ___,

732 – 587 = (___ - ___) + (___ - __) = ___ + __ = ___, etc.

In this case as well as in all previous ones, the coloring of digits and the help with place holders should be progressively omitted.

12. 3. Skill in calculation. In both previous subsections, the method of adding and subtracting was iconically supported and the transformation of expressions (sums and differences) was employed. That makes ground upon which the full understanding of these procedures is gained. By describing these procedures, the following two narrative rules are derived:

Units are added to units (with or without carrying one ten), tens to tens (with or without carrying one hundred) and hundreds to hundreds.

Units are subtracted from units (with or without borrowing one ten), tens from tens (with or without borrowing one hundred) and hundreds from hundreds.

Each of these procedures is followed by a shorthand calculation performed on digits of the involved numbers. In that way formal operations gain a sense as shortened descriptions of more detailed procedures in whose context they are placed. Since the skill of automatic performance of these operations has to be transferred to the cases when the involved numbers are larger than three-digit, children must learn to do them well. To make this task easier attainable by children, an elaboration, consisting of the use of colors and place holders, is set up, indicating place values of digits and the ways of carrying and borrowing. We choose the colors so that the red digits indicate units, the blue tens, and the green hundreds. Let us remark that 17 will denote seventeen units, 17 one ten and seven units, 170 seventeen tens, 170 one hundred and seven tens, etc.

12. 3. 1. Exercises in addition. The first group of exercises relates sums of units, tens and hundreds to their shorthand, decimal notations.

600 + 80 + 5 = $\boxed{6}\boxed{8}\boxed{5}$, 500 + 180 + 5 = $\boxed{\overset{1}{5}}\boxed{8}\boxed{5}$, 400 + 20 + 3 = $\boxed{4}\boxed{2}\boxed{3}$

400 + 10 + 13 = $\boxed{4}\boxed{\overset{1}{1}}\boxed{3}$, 300 + 120 + 3 = $\boxed{\overset{1}{3}}\boxed{2}\boxed{3}$, 300 + 110 + 13 = $\boxed{\overset{1}{3}}\boxed{\overset{1}{1}}\boxed{3}$

This equating of notations should be practiced starting from both sides – the sums are given and children are assigned to fill in the place holders properly and, vice versa, the place holders, which are filled in, are given and children are assigned to write down the corresponding sums.

Now a number of exercises in addition will follow:

$$284 + 579 = \overset{1}{\boxed{7}}\overset{1}{\boxed{5}}\boxed{3} = 863 \qquad\qquad \begin{array}{r} \underline{11}\\ 284\\ \underline{+\,579}\\ 863 \end{array}$$

$$84 + 579 = \boxed{}\boxed{}\boxed{} = \underline{\qquad} \qquad\qquad \begin{array}{r} 84\\ \underline{+\,579} \end{array}\text{, etc.}$$

With the growth of skill, a much larger number of such exercises should be done, without the use of colored digits. For instance:

$$384 + 254 = \boxed{}\boxed{}\boxed{} = \underline{\qquad}\,, \qquad\qquad \begin{array}{r} \overset{-}{3}84\\ \underline{+\,254} \end{array}$$

$$524 + 317 = \boxed{}\boxed{}\boxed{} = \underline{\qquad}\,, \qquad\qquad \begin{array}{r} 5\overset{-}{2}4\\ \underline{+\,317}, \end{array} \quad \text{etc.}$$

Parallel to this addition that is supported by place holders, the vertical addition should run and the technique of forming columns of units, tens and hundreds is easily transferred later to the cases of numbers whose decimal notations have more than three digits.

12. 3. 2. Exercises in subtraction. The first example is taken as a model of elaboration of the shorthand subtraction, followed by the teacher's comments. The teacher says: "When you are doing the difference 624 – 475, you do it this way

$$624 - 475 = (500 - 400) + (110 - 70) + (14 - 5)$$

and you see that 624 has to be decomposed into the sum of 5 hundreds, 11 tens and 14 units, that is

$$500 + 110 + 14 = \overset{1}{\boxed{5}}\overset{1}{\boxed{1}}\boxed{4} \quad .$$

.

Then you subtract units from units: 14 – 5, tens from tens: 11 – 7 and hundreds from hundreds: 5 – 4. You get: 149.

Shortening, you do it this way

$$624 - 475 = \overset{1}{\boxed{5}}\overset{1}{\boxed{1}}\boxed{4} - 475 = 149.$$

Performing vertical subtraction, you do it this way

$$\begin{array}{r} \underline{1\,1} \\ 624 \\ \underline{-\ \ 475} \\ 149 \end{array}$$

As you see 5 units from 14 units are 9 units, 7 tens from 11 tens (1 ten has been borrowed) are 4 tens and 4 hundreds from 5 hundreds (1 hundred has been borrowed) is 1 hundred."

Now a number of examples of shorthand subtraction should be done by children themselves, with and without the help of place holders. Some care has to be taken in examples as the following one

$$603 - 278 = \begin{array}{|c|c|c|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 5 & 0 & 3 \\ \hline \end{array} - 278 = \begin{array}{|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{} \\ \hline 5 & 9 & 3 \\ \hline \end{array} - 278 = 325$$

where the decomposition

$$603 = 590 + 13 = \begin{array}{|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{} \\ \hline 5 & 9 & 3 \\ \hline \end{array},$$

has to be pointed out. As the skill develops, majority of such examples are done without colored digits as well as without the use of place holders.

12. 3.3. Some related remarks. Some teachers report that children, as soon as they have learnt how to perform the shorthand calculations, "forget" to calculate, by decomposing numbers into sums. It is quite normal to expect such an attitude, because these two ways of calculation have two different purposes – one of them is designed to serve as a ground for understanding, and the other one to be a setting for automatic calculation. Of course, a good skill in performing the shorthand calculations could not be taken as a sign of the lack of understanding and by no means should it be blocked from its normal development. To understand a procedure means to know how to reproduce it. But this view should not be taken without any limitation. Children of this age have a natural tendency to do as they are doing, and they are much less inclined to explain what they are doing. In particular, it would be rather demanding to order them to reproduce whole didactical procedures. For the purpose of controlling their degree of understanding, the teacher may use some deliberately programmed exercises as the following:

Fill in what is required

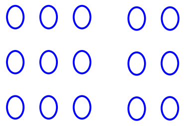
521 – 284 = (___ - 200) + (___ - 80) + (__ - 4), etc.

At the end, by changing slightly the meaning of a well known saying applied to this context, we could say that good actions of children speak louder than words.

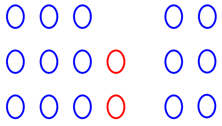
12. 4. Dependence of sums on summands. Procedurally, that is to say, by doing a number of exercises, children are induced to discover and state the rule how a sum changes in dependency on the change of its summands. Let us sketch a programmed exercise that illustrates how it is achieved.

Look at the following picture

(I)

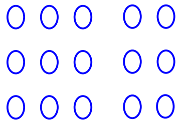


9 + 6

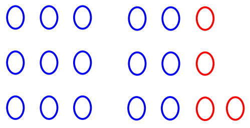


11 + 6

(II)



9 + 6



9 + 10

You can observe that

(i) The first summand has increased by _. The sum was 9 + 6 = __. Now it is 11 + 6 = __. The sum has increased by _.

(ii) The second summand has increased by _. The sum was 9 + 6 = __. Now it is 9 + 10 = __. The sum has increased by _.

When you compare each second arrangement with the first one, you can observe that

(i) The first summand has decreased by _. The sum was 11 + 6 = __. Now it is 9 + 6 = __. The sum has decreased by _.

(ii) The second summand has decreased by _. The sum was 9 +10 = __. Now it is 9 + 6 = __. The sum has decreased by _.

After doing a number of such exercises, the rule of dependence of sum on its summands is stated: *A summand increases (decreases), as much as the sum increases (decreases).*

To make this rule operative, children are assigned to do the exercises as the following ones:

Compare the sums, filling in "<" or ">"

135 + 684 ___ 135 + 729, 135 + 648 ___ 135 + 521

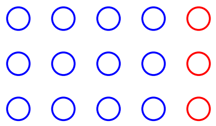
189 + 799 ___ 180 + 799, 189 + 799 ___ 189 + 800, etc.

Without doing the sums, say which of two of them is bigger (smaller) and for how much

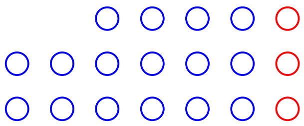
(I) 256 + 385, 249 + 385, (II) 391 + 489, 391 + 495, etc.

12. 5. Dependence of differences on minuend and subtrahend. Procedural elaboration of the dependence of a difference on its minuend and subtrahend will be illustrated again by a couple of examples.

(a) When you compare the differences illustrated by the following arrangements



15 - 3



19 – 3

you can observe that: The minuend has increased by _. The difference was 15 – 3 = __. Now it is 19 – 3 = __. The difference has increased by _.

When you compare the second difference with the first one, you can observe that: The minuend has decreased by _. The difference was 19 – 3 = __. Now it is 15 – 3 = __. The difference has decreased by _.

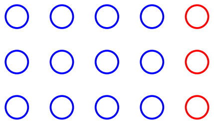
Doing a number of similar exercises, children discover the rule of dependence of difference on its minuend: *A minuend increases (decreases), as much as the difference increases (decreases).*

To use this rule actively, children are assigned to work out examples as the following ones:

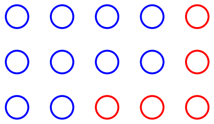
Without finding the differences, find which one of the two is larger (smaller) and for how much:

(I) 725 – 489, 715 – 489, (II) 333 – 188, 353 – 188, etc.

(b) When you compare the first differences with the second one



15 – 3



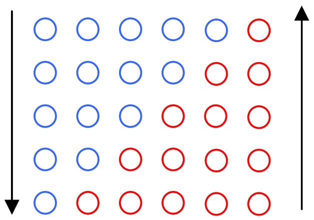
15 – 5

you can observe that the subtrahend has increased by _. The difference was 15 – 3 = __. Now it is 15 – 5 = __. The difference has decreased by _.

When you compare the second difference with the first one, you can observe that: The subtrahend has decreased by _. The difference was 15 – 5 = __. Now it is 15 -3 = __. The difference has increased by _.

After doing a number of similar exercises, the rule of dependence of difference on its subtrahend is stated: *A subtrahend increases (decreases), as much as the difference decreases (increases).*

The fact that an increase (decrease) of the subtrahend causes a decrease (increase) of the difference should also be illustrated itself, because children are apt of making mistakes by relating automatically an increase to an increase and a decrease to a decrease. The following picture is such an illustration.



Looking downwards:
subtrahend increases
difference decreases

Looking upwards:
subtrahend decreases
difference increases

(The illustrated differences are: 6 – 1, 6 – 2, 6 – 3, 6 – 4, 6 – 5).

To also make the last rule operative, exercises similar to those in the previous cases should be assigned to children.

In the case when children are trained well in transformative activities, exercises such as the following ones should be given to them.

(b) The first summand of the sum 357 + 249 increases by 129. Then the sum is

$$(357 + 129) + 249 = (357 + 249) + 129.$$

We see that the sum has increased by 129.

Similarly, in the case of differences, the transformations as these ones

$$(629 - 84) - 218 = (629 - 218) - 84,$$

$$(629 - (218 - 84) = (629 - 218) + 84, \text{ etc.}$$

are used, alternatively replacing the cases of increasing and decreasing.

Such examples can be, of course, programmed by means of place holders. We are inclined to think that these transformative rules and their use with this kind of a motivation, are appropriate to the age of fourth graders.

12. 6. Multiplication by a one-digit multiplier. As it was already said, the block 1 – 1000 is the range of numbers within which the understanding of the main procedures of decimalization is established. And as soon as children have learnt the decimal structure of numbers bigger than 1000, they learn easily by analogy how to extend these procedures to such numbers.

This didactical unit starts with exercises of multiplication by ten.

a) Fill in what is required

(I) 8·10 = __, 80 = _·10, (II) 30 = _·10, 3·10 = __, (III) 10·10 = ___, 100 = 10·__, etc.

(IV) 3·30 = 3·(3·__) = (3·3)·__ = _·10 = __, (V) 4·20 = 4·(_·10) = (4·2)·__ = 8·__ = __,

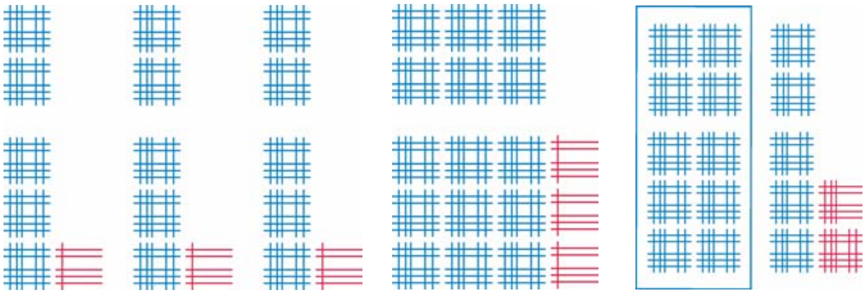
(VI) 7·30 = 7·(3·__) = (7·3)·__ = __·10 = ___, (VII) 8·70 = 8·(7·__) = (8·7)·__ = 56·__ = ___, etc.

The entries in the multiplication table and the product of tens should be coupled together.

b) 8·3 = __, 8·30 = ___; 7·9 = __, 7·90 = ___, etc.

Proceeding gradually, the case of multiplication of two-digit numbers is presented.

c)



3 · 56 = 3 · 50 + 3 · 6 = 150 + 18 = ___

3·56 = 3·(50 + 6) = 3·50 + 3·6 = 150 + 18 = ___.

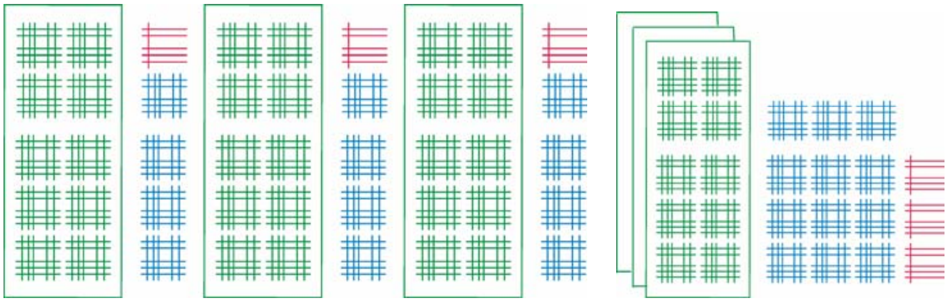
The shorthand form is

$$3\cdot56 = \boxed{1\,|\,\overset{1}{5}\,|\,8} = 168$$

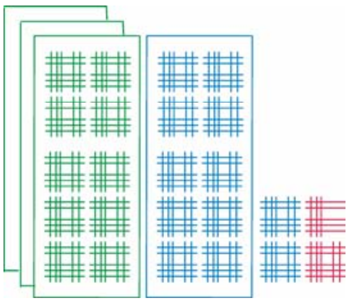
and the teacher's comment runs as follows: We multiply units: 3·6 = 18, getting 1 ten and 8 units; then, we multiply tens: 3·5 = 15. We add tens together: 15 + 1 = 16.

The multiplication of a three-digit number proceeds by representing that number as the sum of hundreds, tens and units. The simplest case is multiplication without any carrying. Then the cases of carrying tens, hundreds and, simultaneously, tens and hundreds follow. We are going to consider only an example that illustrates the last case of these multiplications.

d)



3·146 = 3·100 + 3·40 + 3·6



= 300 + 120 + 18 = 438.

The shorthand calculation is

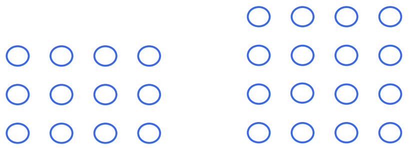
$$3\cdot146 = \begin{array}{|c|c|c|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{} \\ \hline 3 & 2 & 8 \\ \hline \end{array} = 438$$

The teacher’s comment: First we multiply units: 3·6 = 18, getting 1 ten and 8 units. Then we multiply tens: 3·4 = 12, getting 1 hundred and 2 tens and finally, we multiply hundreds: 3·1 = 3. Adding up, tens to tens and hundreds to hundreds, 438 is obtained.

It is essential that children learn well this shorthand way of multiplication that is achieved through performance of a larger number of exercises.

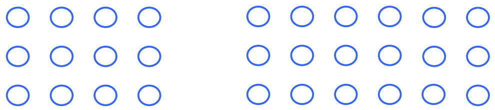
12. 8. Dependence of products on their factors. Using the illustrated examples as, for instance, these ones are

(I)



3 · 4 < 4 · 4

(II)



3 · 4 < 3 · 6

and discussing them with children, the teacher leads his/her class to discover this rule of dependence of product on its factors: *Larger the factors are, larger the product is*.

A more precise dependence of product on its factors is expressed procedurally by means of a number of elaborated exercises.

a) The product 3·4 is given.

(i) The first factor has increased by 2:

$$(3 + 2)\cdot 4 = 3\cdot4 + 2\cdot4,$$

and we see that the product 3·4 has been increased by 2·4 = 8.

(ii) The second factor has increased by 3:

3·(4 + 3) = _·_ + _·_

and we see that the product 3·4 has been increased by _·_ = _.

b) The product 123·5 is given.
(i) The first factor has decreased by 37:

$$(123 – 37)\cdot5 = 123\cdot5 - 37\cdot5,$$

and we see that the product 123·5 has been decreased by __·_ = ___.

(iii) The second factor has decreased by 2:

___·(_ - _) = ___·_ - _·_,

and we see that the product 123·5 has been decreased by ___·_ = ___.

As soon as this procedure has been acquired by children, such exercises should be done without being particularly programmed.

In addition, some corresponding word problems should be assigned. For instance,

Jane and Diana are planting tulips in rows, in each row 16 flowers. Jane has planted 12 rows and Diana 15 rows. How many more flowers has Diana planted? Etc.

12. 9. When we omit the brackets. Composing a triple product, the brackets are used to indicate which two of the three factors are multiplied first. Since this order of multiplication does not affect the value of the product, the brackets are frequently omitted.

When the brackets are placed around products and quotients figuring as components of sums and differences, their role is to indicate that multiplication and division have to be performed first. But these cases are also the instances of expressions when the brackets are omitted. To be aware of this convention of omitting brackets, children have to be trained in leaving them out as well as in bringing them back when the order of calculation has to be indicated. Let us consider some examples that illustrate such an elaboration.

a) When summands are products or quotients, they are usually written without brackets. Omit brackets, where it is possible:

(i) (3·5) + (9·7) = 3·5 + 9·7

(ii) (35·7) + 18 = ____________________________________,

(iii) (81:9) + (14·4) = _________________________________,

(iv) (24 – 11)·9 + (18:3) = _____________________________, etc.

b) When minuends or subtrahends are products or quotients, they are usually written without brackets. Omit brackets, where it is possible:

(i) (24·18) – (71 – 15) = 24·18 – (71 – 15),

(ii) (81·4) – (63:9) = _________________________________,

(iii) (5·38) – (12·4) = _________________________________,

(iv) (721 – 124) – (9:3) = _____________________________, etc.

The following example contains expressions written in the usual way. Children are required to bring back the omitted brackets.

c) Write the brackets to indicate the order in which you will calculate:

(i) $3\cdot18 - 4\cdot6 = (3\cdot18) - (4\cdot6)$,

(ii) $39 - 64:8 =$ ____________________________________,

(iii) $25\cdot4 - 33\cdot3 =$ ____________________________________,

(iv) $(7\cdot9 - 8) - 3\cdot3 =$ ________________________________, etc.

Let us add that the authors of some text-books classify addition and subtraction as operations of lower order and multiplication and division as those of higher order. Then, they state the rule: The operations of higher order are performed before the operations of lower order. Besides lacking a necessary precision, this rule enforces transformations as this one

$$7\cdot(23 - 21) = 161 - 147 = 14,$$

though the way of calculation commanded by brackets is easier:

$$7\cdot(23 - 21) = 7\cdot2 = 14.$$

As it was suggested in the Subsection 10. 3. 2. (The fifth installment of this paper), the expressions as, for instance, $18 - 5 + 3$ should never enter arithmetic at this stage. A clear expression of that kind necessarily combines brackets in one of the following ways $(18 - 5) + 3$ or $18 - (5 + 3)$. Applying repeatedly the cases of omission of brackets stated in the examples under a) and b) (including brackets-free writing of sums and products), all other cases of omission will be covered. For example, the expression

$$3\cdot(19 - 4) - 7\cdot5$$

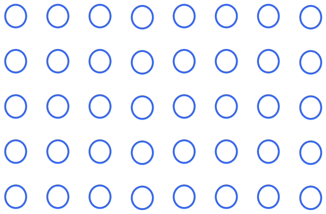
is a difference, its minuend is $3\cdot(19 - 4)$ and its subtrahend is $7\cdot5$. This expression results from the following one

$$(3\cdot(18 - 4)) - (7\cdot5),$$

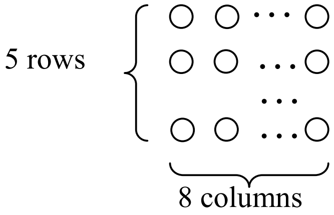
by omitting brackets, according to the case stated in the example under b). At the end, let us say that this procedure of gradual and active omission of brackets is better than an enforcement of children to understand a bombastic rule.

12. 10. Iconic signs with ellipses. In Section 11. 4 (The sixth installment of this paper), the multiplicative schemes were considered as an intuitive base for multiplication (and division). In the case of larger numbers of rows or columns, such signs appear with ellipses that suggest that rows and their elements come one after another. Looking at such designs, the numbers of rows or columns cannot be found by counting them, but they are given as data accompanying such designs. Since these signs represent clearly spacial patterns of sets, children have to learn how to understand the information they convey.

Just to begin with a proper elaboration the teacher could ask children to draw a number of rectangular multiplicative schemes assigning how many rows and columns should each have. As soon as the children have finished drawing such signs, the teacher will represent each of them, using ellipses. For instance, if the scheme

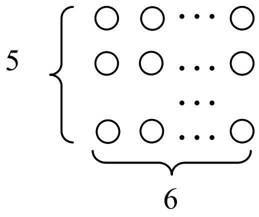


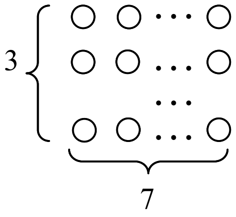
has been drawn, the teacher represents it by the sign

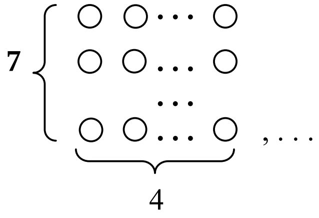


Certainly, a teacher's comment that it is quite fatiguing to draw big schemes would make children to accept more readily this kind of signs.

Reversing this didactical procedure, the teacher will ask the children to draw fully the signs as, for instance,







varying the numbers of rows and columns, but never changing the graphical design of this sign. (Each ellipsis suggests counting: one, two and so on up to an *n*). Teacher's questions as: How many elements in the first row have been drawn and how many left out, How many columns have been drawn and how many left out, etc. should also be useful to train children to use correctly these signs. After a sufficient number of such exercises have been done, the children are ready to use them independently of the teacher's help. Then, they can be assigned the tasks as, for instance, this one:

A plantation of roses in a nursery-garden has 27 rows, in each row 9 seedlings. Drawing seedlings as Ψ, represent that plantation. Calculate the total number of the seedlings.

Let us remark that in this context the word "represent" should not be replaced by the word "draw" A sign with ellipses does not represent a real appearance as a whole, but it is used conventionally to suggest well, essential spacial pattern of that appearance. And if we do not expect that children have a feeling for such finesse, we do expect it from a teacher.

12. 11. Division by a one-digit divisor. Proceeding gradually, the first cases of division will be those when dividends are pure sums of tens. Based on the interdependence of two operations, these cases of division bear resemblance to those in the block up to 100.

These cases of division start with the exercises as the following ones:

You know that	then, you also know that
a) 20·4 = 80	80:4 = 20
b) 70·3 = 210	210:3 = 70
c) 50·9 = ___	450: 9 = ___, etc.

Coloured digits are used to help children discover the formal procedure: The number of tens (blue digits) is divided and then zero is added in writing.

Division of pure hundreds is even easier and it is practiced doing the exercises as, for instance, these are:

d) 200·4 = 800, 800:4 = 200

e) 300·3 = ___, 900:3 = ___, etc.

Here, the formal procedure to be noticed is: The number of hundreds (green digit) is divided and then, two zeros are added in writing.

The corresponding shorthand forms, performed without direct reliance on multiplication, are:

f) $720 : 6 = \overset{1}{\boxed{6\,|\,2\,|\,0}} : 6 = 120$

and, in the form of vertical algorithm,

$$\begin{array}{l} \underline{7}20 : 6 = 120 \\ \underline{12} \\ \quad 0 \end{array}$$

g) $950 : 5 = \overset{4}{\boxed{5}}\boxed{5}\boxed{0} : 5 = 190$

$\underline{9}50 : 5 = 190$
$\underline{45}$
0

Etc.

Now we will consider two cases of division, dependent on the fact whether or not the number of hundreds of the dividend is smaller than the divisor. In the former case, which will be considered first, the dividends will be taken as sums of their tens and units. And then, the rule of division of sums will be applied.

h) 426:6 = (420 + 6):6 = 420:6 + 6:6 = 70 + 1 = 71.

The shorthand forms are:

426:6 = 71

$\underline{42}6:6 = 71.$
6

In the example that follows the number of tens is not divisible by the divisor.

632:8 = (630 + 2):8 = (560 + 70 + 2):8 = (560 + 72):8 = 560:8 +72:8 = 70 + 9 = 79.

At this stage children have already learnt division with remainder and they know how to transform 630 into the sum 560 + 70, transforming similarly in all other examples of that kind.

The second case of division will be illustrated with the following three examples.

i) 675:3 = (600 + 75):3 = (600 + 60 + 15):3 = 600:3 + 60:3 + 15:3

= 200 + 20 +5 = 225.

$675:3 = \boxed{6}\overset{1}{\boxed{6}}\boxed{5} : 3 = 225$

$\underline{6}75:3 = 225$
$\underline{7}$
15

j) $968:4 = (900 + 68):4 = (800 + 168):4 = (800:4 + 168:4 = 200 + 42 = 242.$

$968:4 = \overset{1}{\boxed{8}}\boxed{6}\boxed{8} : 4 = 242$

$$\begin{array}{l} 968:4 = 242. \\ \underline{16} \\ \quad 8 \end{array}$$

k) $875:7 = (800 + 75):7 = (700 + 175):7 = (700 + 140 + 35):7 =$

$= 700:7 + 140:7 + 35:7 = 100 + 20 + 5 = 125$

$875:7 = \overset{1}{\boxed{7}}\boxed{7}\boxed{5} : 7 = \overset{1}{\boxed{7}}\overset{3}{\boxed{4}}\boxed{5} :7 = 125$

$$\begin{array}{l} \underline{8}75:7 = 125. \\ \underline{17} \\ \quad 35 \end{array}$$

12. 12. The word at the end of the way. A great deal of learning arithmetic is carried out within the range of small numbers (up to 1000). Based on this range the meaning and the main properties of the four arithmetic operations are established and the basic procedures of decimalization are elaborated. This knowledge and these skills are carried over to the whole set of natural numbers, constantly placing stress on the analogy with the situations that have already been encountered in the frames of initial blocks. All seven installments of this paper have been written for teachers to help them grasp the stractures of number blocks and to enable them to see clearly how, in a long process of learning, the concepts of arithmetic have been synthesized. By outlining the way of learning that leads from the immidiate observation to iconic and symbolic representations, didactical means and procedures are better appreiciated as a framework within which the pure arithmetic facts are established. Just this framework, and not the formal contents of arithmetic, is something what makes a way of elaboration specific. (The word „broader“ in the title of this paper refers to such a framework).Now we summerize the key features and innovations of our approach.

1. For the sake of a clear comprehension of concepts and following the Skempian way of conception of concepts, we use a tripartite scheme having for its components: corresponding class of examples, mental image and name (including a possible symbolic sign). This scheme supports well the plan of building concepts – concrete examples are gathered together (closely in place and time) and through the activites of their invariable naming or symbolic denoting the corresponding inner representation is formed. Such a scheme is also a useful tool to prevent the tendency of reducing a concept to one of its components or, as it happens often, to its particular examples.

2. The process of abstracting by which the individual concepts of natural numbers are formed, is based on the Cantor Principle of Invariance of Number: Perceiving a set of objects and by „forgetting“ their nature and the way they are grouped and arranged, the pure idea of number is left behind. This is, of course, a cognitive principle which directs and illuminates the process of forming concepts of numbers and operations. The numerical equivalence of sets is instituted by counting, which has its meaning formed up to the upper bound of an already built

block of numbers and ultimately, by attaching the same symbols from the whole set of decimal notations.The use of this principle is a particularly distinguishing feature of our approach to the elaboration of arithmetic.

3. Building and structuring of number blocks with their specific didactical roles, makes a systematic way of exposing and learning arithmetic contents. On the other hand, it is important for a teacher not only to know how individual concepts are formed but also to know their place in the systems that they constitute. The establishment and the active use of properties of operations is characteristic of the learning with understanding and it is a ground for later logical structuring of that kind of knowledge.

4. To understand an activity means to reproduce it at a higher level. In the elaboration of arithmetic, the first step to higher level is achieved when observable things and manipulative activities are presented iconically and the second step is achieved when symbolic codes are attached to iconical representations. A possible loss from the sight of the imaginary border lines between these levels of activities could cause quite a lot of harm to the normal course of abstracting. A careful reader can easily notice as how strictly we respect these border lines, by embodying activities according to the level to which they belong. (As related to the denoting of sets, we called some examples of disrespect of these lines as „barbarisms“).

5. According to the Cantor Principle, grouping of elements of sets does not effect the concept of number. But some specific groupings do effect the way how expressions denoting numbers are composed. Thus, decimal groupings correspond to decimal notations, groupings represented as additive schemes correspond to sums and those as multiplicative schemes to products. As significant signs, the iconic representations of such relevant groupings are treated here with scrupulous attantion to detail. In actual teaching situations, such signs give meaning to the concepts (and, let us avoid to say, define them). For this reason, once fixed, their shape should not be varied throughout the whole course of elaboration of arithmetic.

6. Characteristic for our approach is an intensive use of arithmetic expressions. Contrary to the comprehension of expressions as being stimuli for calculation, we treat them as symbols denoting numbers. In that way we care more about their transformation by the use of established rules of arithmetic. On such skills are, then, based those particular transformations, having decimalization as their ultimate goal. But this programme could not be achieved without a skilful use of place holders which, used in that way, are an invaluable tool for an active elaboration of arithmetic. A proper schedule of lessons as didactical units must also be adapted to this technique of elaboration.For instance, introducing a new concept, its illustrating examples are given first. Then, in such a context, the use of place holders is related to the form of these examples that have been given, making so the requirements clearly determined. As an essential didactical demand, we also emphesize composition of realistic expression only, meaning those supported by the corresponding schemes.

7. Leaving the initial blocks, all other numbers are conceived as sums of products of one-digit numbers and powers of 10 and then, the operations are chiefly performed in the shorthand form. Understanding of such formal procedures is based on analogy with the similar cases from the block 1 – 1000.

Let us add here a remark concerning long division. As a partcularly subtle procedure is the estimation of the quotient digits. Instead of ’’trial and error’’ method, we suggest, for the start, a repeated calculation of the true digit. To illustrate such a direct, ’’increase-by-one’’ method, let us take an unfavourable example:

$$\begin{array}{l r|l}
1001 : 11 = & 9 & 1 \\ \cline{2-3}
\underline{55} & 5 & \\
45 & 2 & \\
\underline{22} & 1 & \\
23 & 1 & \\
\underline{11} & & \\
12 & & \\
\underline{11} & & \\
11 & &
\end{array}$$

Practicing this procedure of finding the true digits in steps, children will be better prepared for some more efficient ways of rounding numbers that they learn later.