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KARAMATA'S PRODUCTS OF TWO COMPLEX NUMBERS

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Abstract. In his paper *Über die Anwendung der komplexen Zahlen in der Elementargeometrie*, Bilten na Društvoto na matematičarite i fizičarite od N.R. Makedonija, kn. I, Skopje, (1950), 55–81 (in Serbian, resume in German) and in the university textbook *Complex Numbers*, Belgrade, 1950, Jovan Karamata (1902–1967) maintains that the role of vector in planimetry can be assigned to complex number through defining and solving certain problems by means of the product $\overline{a}b$ where $\overline{a}$ is the conjugated number of a, whereas a and b are complex numbers which correspond to free vectors $\vec{a}$ and $\vec{b}$. Using geometric interpretation of a and b, Karamata expresses A and B as $A = |a||b|\cos\alpha$ and $B = |a||b|\sin\alpha$. In order to underline the geometric sense of these expressions, Karamata denotes them as $A = (a \perp b)$, resp. $B = (a \mid b)$ designating them "orthogonal product" and "parallel product". By means of these two symbols, considered as products, Karamata interprets those problems in planimetry which correspond to parallelity and orthogonality, and shows how they can be used in deriving Pappus-Pascal and Desargues Theorems.

ZDM Subject Classification: F 54; *AMS Subject Classification*: 01 A 60, 00 A 35

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In his paper *Über die Anwendung der komplexen Zahlen in der Elementargeometrie*, Bulletin de la Société des mathématiciens et des physiciens de la Macédoine, tome I, Skopje, 1950, 55–81 (in Serbian, resume in German) and in the university textbook *Complex Numbers*, Belgrade, 1950 (in Serbian), Serbian mathematician Jovan Karamata (1902–1967) maintains that the role of vector in planimetry can be assigned to complex number through defining and solving certain problems by means of the product $\overline{a}b$ where $\overline{a}$ is the conjugated number of a, whereas a and b are complex numbers which correspond to free vectors $\vec{a}$ and $\vec{b}$. Algebraic manipulation with complex numbers always results in two parts of the product $\overline{a}b = A + Bi$: a real part A, and an imaginary part B, which, by applying geometric interpretation of complex numbers $a = |a|e^{\alpha_1 i}$ and $b = |b|e^{\alpha_2 i}$, Karamata expresses as

$$A = |a||b|\cos\alpha \quad \text{and} \quad B = |a||b|\sin\alpha, \quad \alpha = \alpha_2 - \alpha_1,$$

wherefrom it follows that A is a scalar product, and B is a vector product (considered to be a scalar) of vectors $\vec{a}$ and $\vec{b}$. In order to underline the geometric sense of these expressions, Karamata denotes them as

$$A = (a \perp b) \quad \text{resp.} \quad B = (a \mid b)$$

designating them "orthogonal product" and "parallel product".

A translation of the author's article "*Karamatini proizvodi dva kompleksna broja*" from the book *Metodika i istorija geometrije* (Divčibare, 12–13 oktobra 1996), published by Mathematical Institute SANU in 1997, pp. 31–40.

By means of these symbols it is easy to express the conditions of orthogonality and parallelness for two vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a} \perp \vec{b} \quad \text{if} \quad (a \perp b) = 0 \quad \text{and} \quad \vec{a} \parallel \vec{b} \quad \text{if} \quad (a \mid b) = 0,$$

and since

$$(a \perp b) = (a \mid ib) \quad \text{and} \quad (ia \perp b) = (a \mid b)$$

where i is the imaginary unit, it follows that one product can always be reduced to another one.

Karamata continues with the basic properties of these products which he then uses to algebraically express particular geometric constructions, especially the projective ones. He also uses them to derive some formulae and basic trigonometric propositions. To show simplicity and harmony of particular planimetric propositions, using parallel product for convenience, Karamata gives algebraic interpretations of proofs of two fundamental propositions of projective geometry—Pappus-Pascal and Desargues Theorems.

Of all the properties of Karamata's products we mention here the following: associative law, commutative law—which holds only for orthogonal product

$$(a \perp b) = (b \perp a),$$

whereas in the case of parallel product

$$(a \mid b) = -(b \mid a),$$

—and distributive law for addition, which holds for both products

$$((a+b) \perp c) = (a \perp c) + (b \perp c), \qquad ((a+b) \mid c) = (a \mid c) + (b \mid c).$$

1. Proof of the Pappus-Pascal Theorem

When he was sixteen, Blaise Pascal (1623–1662) wrote a treatise on conic sections (which was lost but was mentioned by Leibnitz), and he also wrote an *Essay on Conics*, in 1640, where he presented the following theorem which is today known under his name:

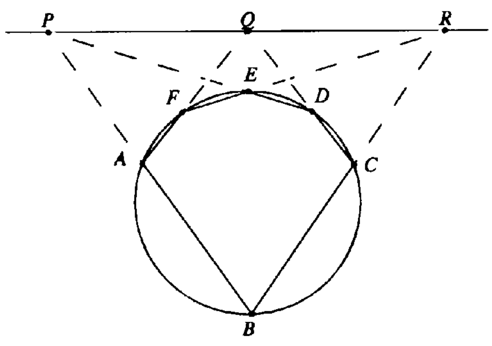

Fig. 1

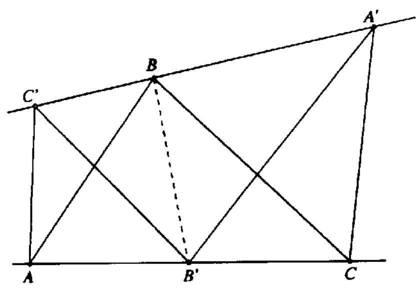

Fig. 2

If a hexagon is inscribed in a conic section, then, when extended, the pairs of its opposite sides will intersect in three collinear points (Fig.1).

If the conic section degenerates into two lines, that is, if the hyperbola degenerates into its asymptotes, then one gets the case described by Pappus, a mathematician of the Alexandrian school at the turn of the 3rd century, which will be discussed in the third part of this paper. In his proof, Karamata considers a special case of the Pappus Theorem (which he named the Pappus-Pascal Theorem) when the vanishing line lies in infinity, which he formulated as (Fig.2):

If the points A, B' and C belong to one line, and points A', B and C' to another, and if

$$AB \parallel A'B' \quad \textit{and} \quad BC \parallel B'C'$$

then $AC' \parallel A'C$ holds true as well.

In order to make his proof well laid out, Karamata divides Figure 2 into two quadrangles (Fig.3), and having used vectors, he indirectly applies translation and congruence.

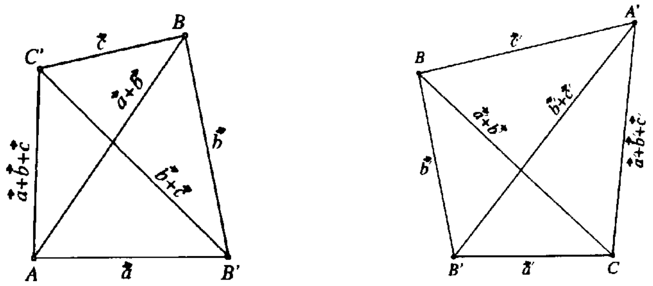

Fig. 3

Thus he reduces the Pappus-Pascal Theorem to the following:

If

$$\vec{a} \parallel \vec{a'}, \quad \vec{b} \parallel \vec{b'}, \quad \vec{c} \parallel \vec{c'},$$

and

$$\vec{a} + \vec{b} \parallel \vec{b'} + \vec{c'}, \quad \vec{b} + \vec{c} \parallel \vec{a'} + \vec{b'},$$

then

$$\vec{a} + \vec{b} + \vec{c} \parallel \vec{a'} + \vec{b'} + \vec{c'},$$

that is, from the parallelness of three homologous sides and non-homologous diagonals, the parallelness of the fourth sides in both considered quadrangles follows.

Expressing this proposition using the parallel product and knowing that

$$(a \mid b) = 0 \quad \text{is equivalent to} \quad \vec{a} \parallel \vec{b},$$

Karamata claims that to prove this proposition, one needs to show that if

$$(a \mid a') = 0, \quad (b \mid b') = 0, \quad (c \mid c') = 0$$

and

$$(a + b \mid b' + c') = 0, \quad (b + c \mid a' + b') = 0,$$

then

$$(a + b + c \mid a' + b' + c') = 0$$

holds true. In his proof he starts from what should be proven, i.e., from $(a+b+c \mid a' + b' + c')$, and by using the assumptions with the property of distribution of the parallel product only, he gets

$$\begin{aligned}(a + b + c \mid a' + b' + c') &= (a \mid a') + (a \mid b') + (a \mid c') \\ &\quad + (b \mid a') + (b \mid b') + (b \mid c') \\ &\quad + (c \mid a') + (c \mid b') + (c \mid c') \\ &= ((a \mid b') + (a \mid c') + (b \mid b') + (b \mid c')) \\ &\quad + ((b \mid a') + (b \mid b') + (c \mid a') + (c \mid b')) \\ &= (a + b' \mid b' + c') + (b + c \mid a' + b') = 0\end{aligned}$$

wherefrom the Pappus-Pascal Theorem directly follows.

2. Proof of the Desargues Theorem

Gérard Desargues (1591–1661), a French engineer and Pascal's friend, published most of his texts in 1639, in the book entitled: *Brouillon project d'une atteinte aux événemens des rencontres du cône avec un plan*, but his principal proposition was printed in 1648, as an appendix to the book *Manière universelle de M. Desargues, pour pratiquer la perspective* of his friend, A. Bosse (1602–1676) who wished to popularize Desargues' practical methods of projective geometry. The proposition reads as follows (Fig. 4):

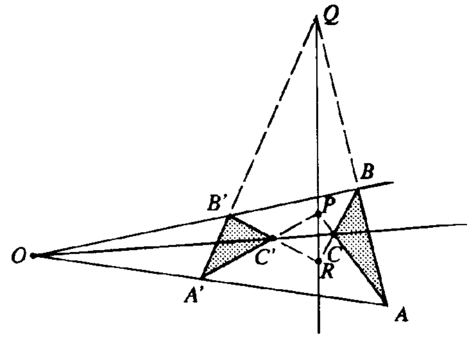

Fig. 4

For two triangles perspective from a point, there holds true that the pairs of homologous sides AB and $A'B'$, BC and $B'C'$, AC and $A'C'$ or their extensions intersect at three collinear points.

Conversely, if the three pairs of homologous sides of two triangles intersect at three points that belong to one line, then the three lines that pass through homologous vertices of these triangles must intersect at one point.

As in the case of the Pappus-Pascal Theorem, Karamata considers a special case of the inverse Desargue Theorem, when the homologous sides of two triangles are parallel, i.e., when the vanishing line lies in infinity, which he formulates in the following way (Fig.5):

If the homologous sides of triangles ABC and $A'B'C'$ are parallel, i.e.

$$AB \parallel A'B', \quad BC \parallel B'C' \quad \text{and} \quad CA \parallel C'A'$$

then the lines passing through homologous vertices AA', BB' and CC' intersect at one point S.

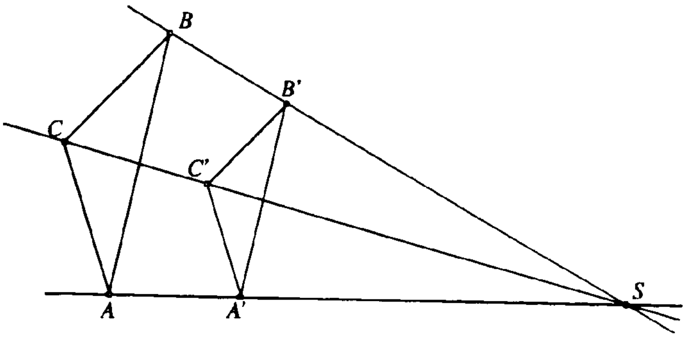

Fig. 5

In order to prove such a formulated theorem, Karamata conveniently divides Fig. 5 into two quadrangles (Fig. 6) whose sides and diagonals are considered as free vectors, and arrives at the following form of Desargues Theorem:

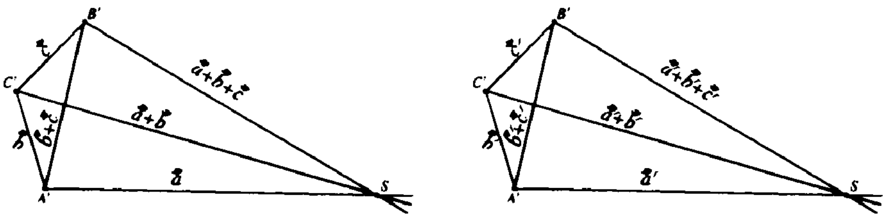

Fig. 6

If

$$\vec{a} \parallel \vec{a'}, \quad \vec{b} \parallel \vec{b'}, \quad \vec{c} \parallel \vec{c'},$$

and

$$\vec{a} + \vec{b} \parallel \vec{a'} + \vec{b'}, \quad \vec{b} + \vec{c} \parallel \vec{b'} + \vec{c'},$$

then

$$\vec{a} + \vec{b} + \vec{c} \parallel \vec{a'} + \vec{b'} + \vec{c'},$$

i.e., the parallelness of three homologous sides and homologous diagonals implies the parallelness of the fourth sides of the quadrangles considered.

Expressed in the symbols of parallel product, the previous theorem reduces to the proof that from

$$(a \mid a') = 0, \quad (b \mid b') = 0, \quad (c \mid c') = 0$$

and

$$(a + b \mid a' + b') = 0, \quad (b + c \mid b' + c') = 0,$$

follow that

$$(a + b + c \mid a' + b' + c') = 0.$$

In the direct proof one gets

$$(a + b + c \mid a' + b' + c') = (a + b \mid a' + b') + (b + c \mid b' + c') + (c + a \mid c' + a'),$$

wherefrom, based on the assumptions that the first two right-hand side sum members equal to 0, follows that, to prove Desargue Theorem, one needs to prove that

$$(c + a \mid c' + a') = 0.$$

As this cannot be proven by using the property of distribution of the parallel product only, Karamata uses two additional properties, from which the relation between the three coplanar vectors

$$(a \mid b)c + (b \mid c)a + (c \mid a)b = 0$$

follows.

The first property claims

$$r(a \mid b) = (ra \mid b) = (a \mid rb), \quad r \in \mathbf{R},$$

whereas the second one claims that every vector $\vec{c}$ can be expressed by its components along the direction of vectors $\vec{a}$ and $\vec{b}$, i.e., there exist two real numbers q and r such that $\vec{c} = q\vec{a} + r\vec{b}$ for every $(a \mid b) \neq 0$.

To prove that $(c + a \mid c' + a') = 0$, Karamata takes arbitrary real numbers p, q and r

$$p = (b \mid c), \quad q = (c \mid a), \quad r = (a \mid b),$$

and finally arrives at

$$\begin{aligned}(pa + qb + rc \mid pa' + qb' + rc') \\ = pq(a + b \mid a' + b') + qr(b + c \mid b' + c') + rp(c + a \mid c' + a').\end{aligned}$$

However, since for such selected p, q and r, the left-hand side and the first two sum members of the right-hand side equal to 0, it follows that

$$rp(c + a \mid c' + a') = 0,$$

and since $r \neq 0$ and $p \neq 0$ it follows

$$(c + a \mid c' + a') = 0,$$

which directly proves the Desargues Theorem.

But Karamata shows that, by introducing an intermediate quadrangle, i.e., the three auxiliary vectors $\vec{a''}$, $\vec{b''}$ and $\vec{c''}$, Desargue Theorem could be derived by two consecutive applications of Pappus-Pascal Theorem. Namely, if the auxiliary vectors are chosen so that, with regard to vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, they satisfy the Pappus-Pascal Theorem (Fig.7), i.e., that

$$(a \mid a'') = 0, \quad (b \mid b'') = 0, \quad (c \mid c'') = 0,$$

and

$$(a + b \mid b'' + c'') = 0, \quad (b + c \mid a'' + b'') = 0,$$

then

$$(a + b + c \mid a'' + b'' + c'') = 0.$$

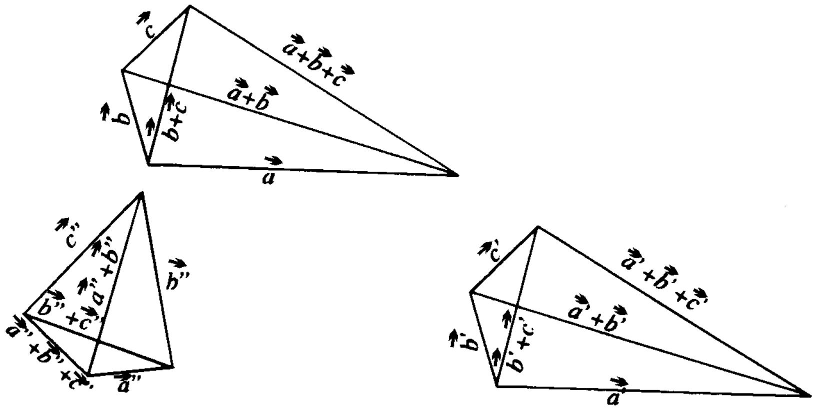

Fig. 7

However

$$(a'' \mid a') = 0 \quad \text{since} \quad (a'' \mid a) = 0 \quad \text{and} \quad (a \mid a') = 0,$$

so by analogous reasoning one can claim that

$$(b'' \mid b') = 0, \qquad\qquad (c'' \mid c') = 0,$$
$$(b'' + c'' \mid a' + b') = 0, \qquad (a'' + b'' \mid b' + c') = 0,$$

and from the second application of the Pappus-Pascal Theorem—this time to vectors $\vec{a'}$, $\vec{b'}$ and $\vec{c'}$, it follows that

$$(a'' + b'' + c'' \mid a' + b' + c') = 0,$$

wherefrom directly follows that

$$(a + b + c \mid a' + b' + c') = 0,$$

which is the Desargue Theorem.

3. Generalization of the parallel product

In his proof of the Pappus-Pascal Theorem Karamata only uses the distribution law for the parallel product. In order to generalize it, with the aim of proving the general case of the theorem, he consideres two pairs of points, A, B and A', B', to which he unambiguously maps a real number Φ in the form of a symbolic product

$$\Phi = \Phi(AB, A'B'),$$

for which an analogue of the distribution law and the sense of the number 0 are defined.

In order to define the analogue of distribution law, Karamata introduces yet another point C, such that

$$\Phi(AB, A'B') = \Phi(AC, A'C') + \Phi(CB, C'B'),$$

where AB is considered to be a certain "punctual" sum of A and C, which is neither ordinary nor vector, what Karmata denotes by

$$AB = AC \widehat{+} CB,$$

so that the distribution law takes the following form

$$\Phi(AC \widehat{+} CB, A'B') = \Phi(AC, A'C') + \Phi(CB, C'B').$$

Number 0 corresponds to those pairs of points AB and $A'B'$ for which

$$\Phi(AB, A'B') = 0,$$

in case when lines AB and $A'B'$ intersect at the vanishing point or when all of the four points lie on the same line. As an arbitrary line h can be taken for the vanishing line (Fig.8), Karamata introduces $(AB, A'B')_h$ as a shorter notation for $\Phi(AB, A'B')$.

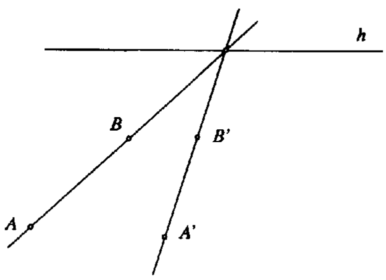

Fig. 8

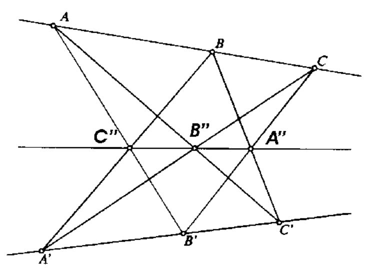

Fig. 9

Karamata then maintains that if the previously introduced distribution law and number 0 are taken for axioms, then those axioms and the axioms of incidence constitute projective geometry, because it can be shown that the Pappus Theorem, from which Desargues Theorem can be derived, holds true, which, in turn, implies the basic proposition of projective geometry concerning the invariance of double ratio. This is why he proves the general Pappus Theorem by using the newly introduced symbolic product. This general proposition reads (Fig.9):

Let points A, B and C lie on one line, and points A', B' and C' lie on another. When these points are connected with polygonal line $AB' : CA' : BC' : A$, then the intersection point C'' of lines AB' and BA', intersection point B'' of lines AC' and CA' and intersection point A'' of lines BC' and CB' are collinear.

In his proof, Karamata draws the vanishing line h through points A'' and C''. To prove that point B'' also lies on that line, i.e., that the lines AC' and CA' also intersect at the vanishing point, he shows that from

$$(AB', BA')_h = 0 \quad \text{and} \quad (BC', CB')_h = 0$$

it follows

$$(AC', CA')_h = 0.$$

Here he uses the distribution law for this product, the definition of 0 and the fact that

$$AC' = AB \hat{+} BB' \hat{+} B'C' \quad \text{and} \quad CA' = CB \hat{+} BB' \hat{+} B'A'.$$

At the end of his paper *Neke primene kompleksnog broja u elementarnoj geometriji* (Some applications of complex number in elementary geometry), Karamata expresses symbolic product Φ through parallel product and complex functions, claiming that it would be interesting to see how the same method can be used to get Pascal Theorem regarding conic sections.

As Karamata himself notes in the paper, the terminology used in the formulation and proof of the Pappus-Pascal and Desargues Theorems implies that it is possible to prove Desargues Theorem for a planar case using all 8 projective axioms of incidence. On the other hand, the proof cannot be completed with just the first five axioms of incidence. If those 5 axioms are complemented with the Pappus Theorem, which is attributed the role of axiom, it is possible to prove Desargues Theorem, which G. Hassenberg showed in 1905 (*Beweis des Desarguesschen Satzes aus dem Pascalischen*, Mathematische Annalen, 61). Also notable in these geometry-related papers is Karamata's clear methodological approach to mathematical proofs. Namely, he begins with the special case of the theorem to be proved, and then generalizes the elements used in his proof. Because his proof that the vanishing line lies in infinity was predominantly based on the distribution law of parallel product, Karamata claims the following: *Indeed, if we want to get rid of parallelness in the proof, instead of the line that lies in infinity, we should suppose that the vanishing line lies in finiteness, and interpret the whole proof projectively . . . In essence, this proof features the concept of parallel product that is manifested* *in the form of the coordination between the set of four points and the set of real numbers, for which the analogue of distributive law holds true, disappearing when the lines defined by these points are parallel, i.e., when they intersect in infinity.*

The originality of Karamata's approach to the topic, coupled with the simplicity and elegance of the proofs exposed, speaks best not only in favor of his exquisite gift for mathematics, but also of his wide mathematical knowledge and his versatile mathematical interest. One must not forget that there are two results from disparate areas of mathematics that earned him fame amongst his peers—the first result concerns the theory of divergent series—more precisely, inverse Tauberian Theorems—whereas the second one deals with the theory of regularly varying functions. However, his papers and results in geometry were somewhat unjustly overlooked. For example, in his paper entitled *Eine elementare Herleitung des Desarguesschen Satzes aus dem Satze von Pappos-Pascal*, Elemente des Mathematik, 5, 1950, 9–10 (in German) Karamata gave planimetric proof of Desargue Theorem by applying the Pappus-Pascal Theorem three times.

Let us add that Karamata served as the editor-in-chief of *L'Enseignement Mathématique* whose centennial anniversary was celebrated in 2000 (see `http://www.unige.ch/math/EnsMath/`).

For the reader interested in details of Karamata's work and life, see [A. Nikolić, *Jovan Karamata (1902–1967)*, Novi Sad J. Math., Vol. 32, No. 1, 2002, 1–5], available also on the Internet.

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