THE TEACHING OF MATHEMATICS
2005, Vol. VIII, 1, pp. 1–14

ON POLYGONS AND POLYHEDRA

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Abstract. This paper is written for those Gymnasium students with an intensive interest in mathematics who are often in search for some extra reading matters not being on school curriculum. We have chosen to elaborate here a number of interesting and beautiful geometric topics: the diagonals of polygons and the triangulations, the art gallery theorems, the Euler formula, regular polyhedra, that could satisfy their curiosity and instigate their further reading.

ZDM Subject Classification: G44; *AMS Subject Classification*: 00 A 35

Key words and phrases: Diagonals of polygons, triangulations, art gallery theorems, Euler formula, regular polyhedra.

1. Introduction

This text is addressed primarily to the talented Gymnasium students and to their teachers. However, we do hope that a university student interested in geometry may also find some parts of this text interesting. Polygons and polyhedra are classical and very important objects both in mathematics and in sciences in general. We are particularly interested in the interplay of geometrical and combinatorial considerations and properties of these objects.

There are many other themes that are convenient for illustration of the interplay of geometry and combinatorics such as:

- simplicial complexes and their connections with the partially ordered sets (the order complex) and with the families of sets (the nerve of the family);
- the Helly's theorem and many related results (the theorems of Caratheodory, Radon, Tveberg, Jung, Krasnoselski);
- integer lattice polygons - the Pick's formula, the Minkowski's theorem, the Ehrhart's polynomial, Newton's polyhedra;
- the characterization of the planarity of the graphs (the Kuratowski's theorem).

Some of these themes require a little bit more advanced tools for their treatment. All of them are extremely attractive and, though classical, they are still a good motivation for further research. As an example, let us mention a far reaching extension of the Helly's theorem, the characterization of f-vectors of the simplicial complexes which arise as the nerve of a family of convex sets in $\mathbf{R}^d$, by Gil Kalai in 1984.

The plenary lecture delivered at the Annual Seminar of the Mathematical Society of Serbia.

The reader interested to learn more about these topics and about many other mathematical themes, is encouraged to visit the web site of the American Mathematical Society and its feature column (`http://www.ams.org/featurecolumn/archive/`).

2. Polygons

The notion of the convex planar polygon is a simple one that can be described in an elementary way. It is a little bit more complicated to introduce the notion of the (non necessarily convex) planar polygon. We start by defining the simple closed polygonal line.

Definition 2.1. The simple closed polygonal line in the plane is a finite family of intervals in that plane $[v_1, v_2], [v_2, v_3], \dots, [v_{n-1}, v_n], [v_n, v_1]$ so that the only two consecutive of them intersect in the common vertex. (We consider also the intervals $[v_n, v_1]$ and $[v_1, v_2]$ to be consecutive.)

It could be proved by elementary methods that a simple closed planar polygonal line divides the plane into two parts, an inside and an outside. The former is bounded and is *a simple polygon* with vertices $v_1, v_2, \dots, v_n$ and edges $[v_1, v_2], [v_2, v_3], \dots, [v_{n-1}, v_n], [v_n, v_1]$. Of course, this fact is a special case of the Jordan curve theorem, which claims the same for any simple closed curve in the plane. Its proof is not elementary and it requires some topological considerations.

2.1. Diagonals and triangulation

A segment joining two non-consecutive vertices of a polygon is its diagonal. The diagonal which lies totally in the interior of the polygon is its interior or internal diagonal. One of the reasons for the importance of interior diagonals comes from the fact that they could be used (if they exist) to triangulate the polygon. However, for a very convoluted non-convex polygon it is not clear whether an interior diagonal exists.

The following theorem shows this to be the case.

Theorem 2.2. *Every planar simple n-gon P with $n \geq 4$, has at least one internal diagonal.*

Proof. Let us consider a vertex v_j of P so that the angle of P at v_j is convex, i.e. $\alpha_j < \pi$. Such a vertex always exists since otherwise we would be considering the angle in the exterior of the polygonal line and not in its interior. If no edge of the polygon crosses the diagonal $[v_{j-1}, v_{j+1}]$, this is an internal diagonal the existence of which was to be proved. (Here and to the end we assume the addition mod n, i.e. $v_{n+1} = v_1$.)

In the other case, let v_i be the vertex of P inside the triangle Δ_j with vertices v_{j-1}, v_j, v_{j+1}, more precisely the first one the ray $[v_j, v_{j-1})$ would meet when rotating around the vertex v_j towards the vertex v_{j+1}. If the diagonal $[v_j, v_i]$ is not

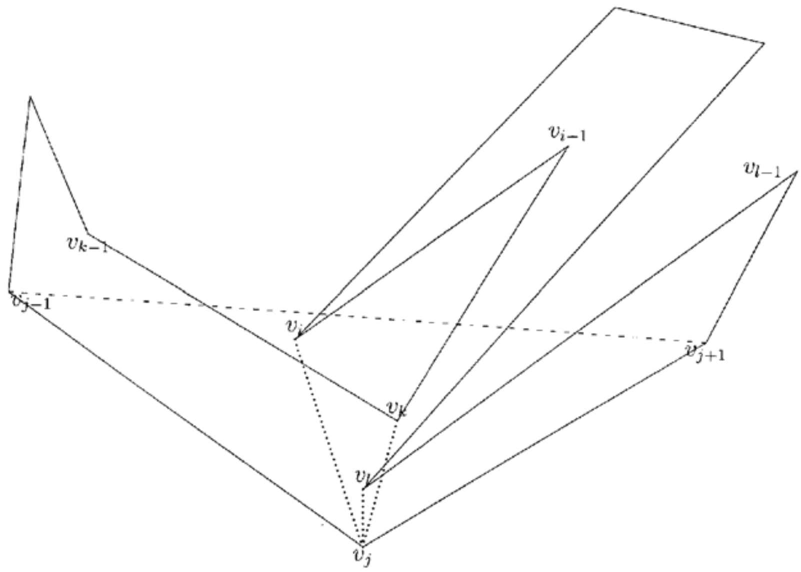

Fig. 1

crossed by any edge of P, it is an internal diagonal. In the other case, let $[v_{k-1}, v_k]$ be the edge of P crossing the diagonal $[v_j, v_i]$ in the point nearest to v_j, and let (for example) v_k be inside the triangle Δ_j. Again, if the diagonal $[v_j, v_k]$ is not crossed by any edge of P, it is an internal diagonal. In the other case, let $[v_{l-1}, v_l]$ be the edge of P crossing the diagonal $[v_j, v_k]$ in the point nearest to the vertex v_j, and let (for example) the vertex v_l be between the rays $[v_j, v_i)$ and $[v_j, v_k)$. Once more, if the diagonal $[v_j, v_l]$ is not crossed by any edge of P, it is an internal diagonal. If some edges of P cross the diagonal $[v_j, v_l]$, they have to be contained between the rays $[v_j, v_i)$ and $[v_j, v_k)$. Let v_m be the first endpoint of these edges which the ray $[v_j, v_k)$ would meet while rotating around the point v_j towards the point v_i. It is easy to check that no edge of P could cross the diagonal $[v_j, v_m]$, and this is an internal diagonal of P. ■

The main consequences of the above result is the fact that every simple planar polygon could be triangulated without adding new vertices to the triangulation. Namely, we prove:

Corollary 2.3. *Every planar simple polygon could be triangulated by the internal diagonals.*

Proof. Let us consider a simple n-gon. If $n \geq 4$, there is an internal diagonal of P, and it divides P into two polygons, a k-gon P' and an l-gon P''. Then $k + l = n + 2$ and since $k, l \geq 3$, each of these polygons has strictly fewer vertices than P, i.e. $k, l \leq n - 1$. Now, if some of the polygons P' and P'' has more than 3 vertices, it has an internal diagonal dividing it into two polygons with a fewer number of vertices, and we continue the procedure. In each step we obtain the polygons with a fewer number of vertices and edges. We end when none of the polygons in which P is subdivided admits an internal diagonal. According to the previous theorem, this happens when all these polygons are triangles. ∎

The triangulations are the combinatorial tool which enables us to treat the geometrical questions more successfully. One of the results which uses this fact is an important theorem saying that, if two polygons have the same area, one of them could be cut up into a finite number of pieces which could be reassembled to form the other polygon.

It is easy to determine the number of triangles into which an n-gon is triangulated by the internal diagonals, and the number of the internal diagonals required to do that.

THEOREM 2.4. *Every triangulation of an n-gon P has exactly $n-2$ triangles and requires exactly $n-3$ internal diagonals.*

Proof. Let us denote with d the number of the internal diagonals in any triangulation of an n-gon. Since an internal diagonal divides a region into two regions, by drawing any internal diagonal we increase a number of polygons by 1. So, an n-gon triangulated by d internal diagonals, will be subdivided into $d+1$ triangles.

These triangles will have totally $3(d+1)$ edges. The edges of the triangles will be the edges of the n-gon (each counted once) and the internal diagonals (each counted twice). So, we obtain the equation $3(d+1) = n + 2d$. By solving this equation, we get $d = n-3$, i.e. there are $n-3$ internal diagonals drawn in any triangulation of an n-gon, and the triangulation consists of $n-2$ triangles. ∎

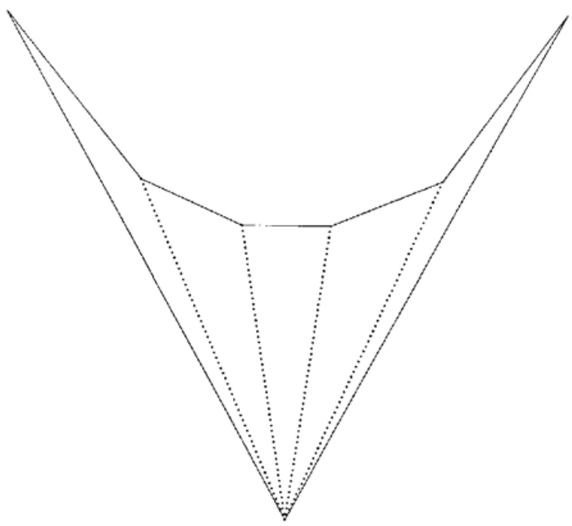

Fig. 2

It follows that every n-gon has at least $n-3$ internal diagonals. There are also examples of n-gons showing that this estimate is the best possible, i.e. having exactly $n-3$ internal diagonals.

The example of such 7-gon is in Figure 2, and it is obvious how to modify it to obtain the example of an n-gon with exactly $n-3$ internal diagonals. (All the vertices except one are on a convex curve, and the remaining vertex is below that curve.)

It is worth noting that the previous theorem implies that a simple n-gon having exactly $n-3$ internal diagonals admits the unique triangulation by the internal diagonals.

2.2. Ears

Let P be a simple planar polygon and let us consider one of its triangulations. The triangles of this triangulation could have 1, 2 or 3 internal diagonals of P as their edges. The triangles having 1 internal diagonal and 2 edges of P for its edges will be called the *ear triangles* of the triangulation. The common vertex of two edges of P which are the edges of an ear triangle, will be called an *ear vertex* of the triangulation.

The question arises whether each triangulation of a simple n-gon has an ear triangle. The answer is given by the following theorem.

THEOREM 2.5. *There are at least two ear triangles in each triangulation of a simple polygon.*

Proof. Let us consider a triangulation of an n-gon and let us denote the number of the ear triangles with a, the number of the triangles whose two edges are the internal diagonals of the n-gon with b, and the number of the triangles whose all three edges are the internal diagonals of the n-gon with c.

By Theorem 2.4 we have the equality $a+b+c=n-2$.

Every edge of the n-gon is the edge of exactly one triangle of the triangulation. So, by counting the edges of the n-gon in all the triangles of the triangulation, we obtain another equality $2a+b=n$.

By subtracting these two equalities, one from the other, we obtain the equality $a-c=2$, i.e. $a=c+2$. So, the number of the ear triangles is greater by 2 than the number of the triangles whose all edges are the internal diagonals. Consequently, there are at least 2 ear triangles in each triangulation of a simple polygon. ■

The example given in Figure 2 is also the example of a simple polygon whose unique triangulation has exactly two ear triangles. As we noticed earlier, this example could be easily modified to produce the polygons with any number of edges having exactly 2 ear triangles in its unique triangulation.

Now we use the above theorem to show the property of the triangulation of a simple polygon which we will need in the following subsection.

THEOREM 2.6. *For each simple polygon and each of its triangulations, the vertices of the polygon could be colored by three colors so that every triangle of the triangulation has the vertices colored by different colors.*

Proof. We prove the theorem by induction on n—the number of vertices of the polygon. For $n = 3$ the statement of the theorem is trivially true.

Let us suppose the theorem to be true for any simple polygon with $n - 1$ vertices, and consider any n-gon P and any of its triangulations. By the previous theorem, this triangulation has an ear triangle with the ear vertex v_j. If we remove this ear triangle from the polygon P, we obtain a simple polygon P' with $n - 1$ vertices $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n$, and its triangulation induced by the triangulation of P.

By the induction hypothesis, the vertices of P' could be colored by three colors so that the triangles of its triangulation are colored by three different colors. If we consider now the polygon P, we see that all of its vertices are colored except for the vertex v_j. The vertex v_j is a vertex of only one triangle of the triangulation of P (namely, the removed ear triangle). The remaining two vertices of this triangle (v_{j-1} and v_{j+1}) are already colored by two colors, and we could assign the third color to v_j, and obtain the proper coloring of P. ■

2.3. Art gallery theorems

The following question was posed by Victor Klee in 1973.

QUESTION. *What is the number of points (called guards) in a simple planar n-gon needed to "guard" the entire n-gon?*

Here, any point x guards all the points of the n-gon P which are visible from x through P, i.e. the points $y \in P$ satisfying $[x, y] \subseteq P$. The terminology is based on the interpretation of the simple polygon as an art gallery or museum and we ask for the minimal number of points (occupied by guards) from which all the points of the polygon are visible.

The same question could be posed in the terms of illumination of the polygonal region, asking about the minimal number of lamps needed to illuminate a region which has a form of a simple planar polygon.

Before we proceed, let us formulate an exercise for the interested reader who wants to get a feeling about possible answers to the above question.

EXERCISE 2.7. What is the minimal number of vertices (and edges) a simple planar polygon has to have in order to require 2 guarding points, or (a little bit more difficult) 3 guarding points?

In the following Figure 3 we show a 12-gon which requires 4 guarding points.

Namely, no two "peak" vertices in this figure could be guarded from the same point, and since there are 4 peak vertices, at least 4 guards are required. It is obvious how to generalize this example to the example of a $3k$-gon requiring k

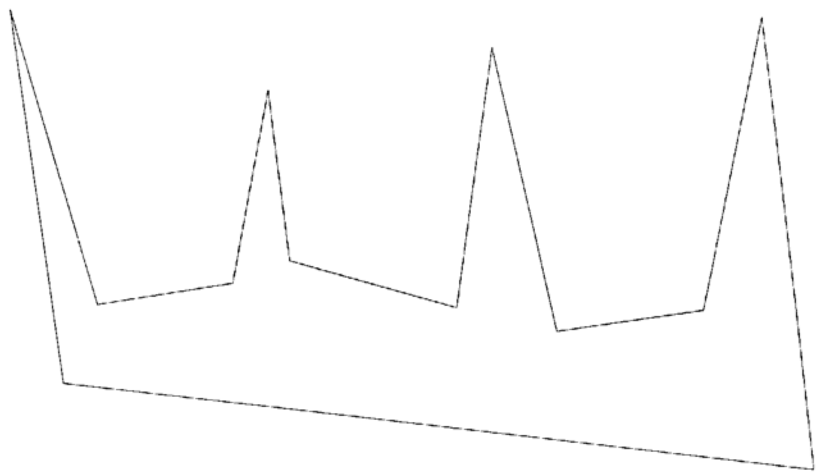

Fig. 3

guards. So, it follows that $\lfloor \frac{n}{3} \rfloor$ guards are necessary for some n-gons. There was a conjecture that $\lfloor \frac{n}{3} \rfloor$ guards are sufficient for every n-gon. V. Chvátal was the first to answer the Klee's question by proving this conjecture in 1975.

Here we present the theorem giving the complete answer to the Klee's question, with another proof due to S. Fisk (in 1978), which deserves our attention because of its elegance and simplicity.

THEOREM 2.8. $\left\lfloor \dfrac{n}{3} \right\rfloor$ *guards suffice to guard any simple planar n-gon.*

Proof. Let us consider some simple planar n-gon P and one of its triangulation. By Theorem 2.6, the vertices of P could be colored by three colors so that every triangle of the triangulation has the vertices colored by different colors. Certainly, vertices of one color guard all the triangles, and thus they guard the entire polygon P. If we choose the color assigned to the smallest number of vertices, we get no more than $\lfloor \frac{n}{3} \rfloor$ guards. ■

If the reader didn't succeed to answer the Exercise 1, she/he could now find the answer easily.

It should be noted that all guards are placed at vertices of the polygon. For some polygons the number of guards placed in any point of the polygon could be smaller than the number of guards placed at vertices.

This question was generalized in a number of different ways. One related problem was to determine the minimal number of guards that suffice for the art gallery with n walls and h holes (the polygon with h polygonal "holes" with the total of n edges). Note that such a region is not bounded by a simple polygonal line, but by several disjoint polygonal lines. It is proved that $\lfloor \frac{n+h}{3} \rfloor$ guards suffice in that case.

Another related question is to consider some special types of polygons, and to determine whether the minimal number of guards could be smaller for such polygons. E.g. several authors considered the problem of guarding the orthogonal art gallery whose all walls are parallel to one of the two orthogonal lines in the plane. It was proved that $\lfloor \frac{n}{4} \rfloor$ guards are sometimes necessary and always sufficient to guard such an n-gon.

3. Polyhedra

The notion of the polyhedron (in dimensions 3 and greater) is significantly more complicated than that of polygon, and it would require some tiresome clarification to give a precise definition and explain its meaning. Instead of doing so, we choose to restrict our considerations to the case of the bounded convex polyhedron which is already sufficiently interesting and in which case we also obtain meaningful and highly non-trivial results.

Definition 3.1. The convex polyhedron in $\mathbf{R}^d$ is the intersection of finitely many half-spaces. The convex polytope is the convex hull of finitely many points.

It could be proved that the notions of a bounded convex polyhedron and a convex polytope coincide in $\mathbf{R}^d$. We will also restrict our consideration to the dimension $d = 3$. The results could be generalized to higher dimensions, but the proofs would require more complicated techniques.

The polyhedra are classical, and certainly very important and interesting object both in mathematics and in science, and there are many interesting facts about them not covered in this article. We suggest the interested reader to consult the books [2] and [3] as the starting point in her/his journey throughout the world of polyhedra.

3.1. Graph of the polyhedron

Let us imagine the watching of a bounded convex polyhedron from a point outside of it, placed below some 2-dimensional face, but very close to it. If this face is transparent (e.g. made of glass), we would be able to see all the other faces (and edges) of the polyhedron through this one.

To describe it a bit more correctly, let x be a point outside the polyhedron P, very close to some interior point of the 2-dimensional face F of P (so close that the interval joining the point x to any vertex v of P which is not a vertex of F, crosses F in an interior point). Let π_F be the radial projection (with the center at the point x) to the face F. The projections of vertices and edges of P form a planar graph $G(P)$, which we call *the graph of the polyhedron P*. The projections of the remaining 2-dimensional faces of P form a dissection of F in finitely many polygons. In the Figure 4 below, we present the graphs of the cube and the tetrahedron.

It could be proved that (besides its planarity), the graph $G(P)$ is also always 3-connected, meaning that any two of its vertices could be joined by three paths,

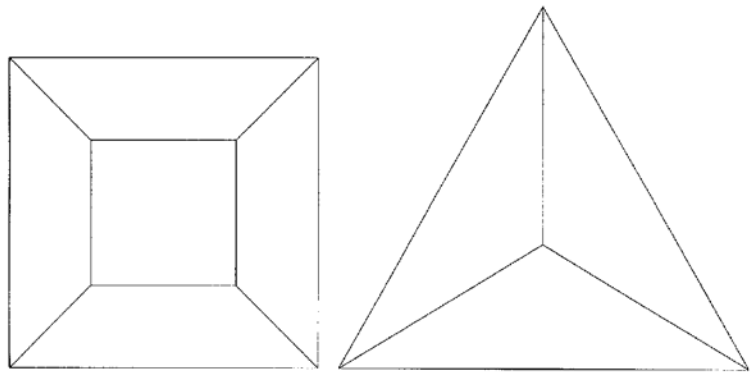

Fig. 4

any two of which meet only at their end-points. The opposite is also true, and this is the famous Steinitz's theorem. Its proof is beyond the scope of this article.

THEOREM 3.2. (Steinitz's theorem) *A graph is isomorphic to a graph of a* 3-*dimensional polyhedron if and only if it is planar and* 3-*connected.*

The notion of the graph of a polyhedron (and the Steinitz's theorem in particular) enables one to study 3-dimensional polyhedra using their graphs and their properties. In the remaining part of this subsection, we define a type of graphs (namely trees), and prove one of their properties needed in the next subsection. A finite sequence of different edges of a graph is called a cycle, if the end-point of any edge is the starting-point of the next edge (including that the end-point of the last edge is the starting-point of the first edge). The property of any cycle is that it bounds a region in the plane (a polygonal region).

DEFINITION 3.3. A connected graph is called a tree if it contains no cycles.

The obvious property of a tree is that it does not bound any region in the plane, and vice-versa if a connected graph doesn't bound any region, then it doesn't contain any cycle and it is a tree.

THEOREM 3.4. *If a tree has v vertices and e edges, than $v = e + 1$.*

Proof. We can build up a tree starting from one vertex by adding one edge (whose one vertex belongs to the existing graph) and one vertex (the other vertex of the added edge) in each step. It is obvious that the graph obtained after each step is connected. Since the tree has no cycles, in no step we could add an edge whose both vertices belong to the existing connected graph. Also, since the tree is a connected graph, we can build up the whole tree in such way. We started with 1 vertex and no edges, and in each step we add 1 vertex and 1 edge. So, we get $v - e = 1$. ■

3.2. Euler formula

The most important property of polyhedra, not only among those mentioned in this article, is described by the *Euler formula*, which determines the value of the Euler characteristic. We start by giving a necessary definition.

Definition 3.5. For the bounded convex 3-dimensional polyhedron P, let v be the number of its vertices, e the number of its edges and f the number of its faces (of dimension 2). Then Euler characteristic of P is the number $v - e + f$.

Now we are ready to formulate and prove the main theorem—well known Euler formula.

Theorem 3.6. (Euler formula) *The numbers of vertices, edges and faces of any bounded convex* 3*-dimensional polyhedron satisfy the relation* $v - e + f = 2$*.*

Proof. Let us consider the graph of the polyhedron P. Let us also imagine the exterior of the graph to be the ocean, the edges of the graph to be dikes made of earth, and the faces bounded by the cycles of the graph to be the dry fields. When the dike represented by the edge adjacent to the ocean is "breached", the field on the other side of the dike is opened up to the ocean and becomes flooded. This means that we could decrease the number of faces (bounded regions) of the graph by 1, by removing 1 edge of the graph. Since the faces or bounded regions of the graph are the projections of the faces of P (except for the face F), there are $f - 1$ faces of the graph. So, by removing $f - 1$ edges of the graph, we obtain the new connected graph with no cycles (therefore a tree), and with v vertices.

In the Figure 5 below, we show one possible way to remove some edges of the graph to obtain connected graph with no cycles, at the example of the graph of the octahedron. In this figure the removed edges are represented as dashed intervals, and the remaining edges as "normal" intervals.

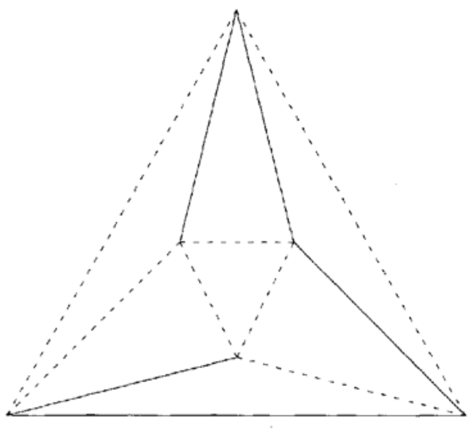

Fig. 5

Since this graph is a tree, the number of its edges (non-removed edges of the original graph) is $v-1$ (by Theorem 3.4). So, the total number of edges ("normal" and "dashed") is $e = v - 1 + f - 1 = v + f - 2$. The Euler formula follows. ■

In the following subsections we show how the Euler formula could be used to obtain many other results.

3.3. Descartes's lost theorem

Euler formulated his formula in 1750, and provided a proof for it a few years later. His proof was not completely correct by modern standards for a proof, but his idea and technique could be used to prove the formula. It is interesting that much earlier Descartes discovered a theorem, which would have enabled him to deduce the Euler formula. However, he didn't do so. He wrote an *Elementary Treatise on Polyhedra* around 1620, but this Treatise was never published. His theorem from this Treatise is usually called the Descartes's lost theorem.

In this article we will go the other way around, and use the Euler formula to deduce the Descartes's lost theorem.

THEOREM 3.7. (Descartes's lost theorem) *The sum of all interior angles in all the faces of the bounded convex* 3*-dimensional polyhedron* P *with* v *vertices equals* $2\pi \cdot v - 4\pi$.

Proof. For every $k \geq 3$, let us denote with p_k the number of the faces of P which are k-gons. Then the sum $\sum_{k\geq 3} kp_k$ represents the sum of number of all edges in all the faces. Since every edge belongs to exactly 2 faces, it is counted twice in this sum, and we have $\sum_{k\geq 3} kp_k = 2e$, where as usually e denotes the number of edges of P.

The sum of all interior angles in any k-gon is $(k-2)\pi$. Therefore the sum of all interior angles in all the faces is:

$$\begin{aligned}\sum_{k\geq 3} p_k \cdot (k-2)\pi &= \pi\Big(\sum_{k\geq 3} kp_k - \sum_{k\geq 3} 2p_k\Big)\\ &= \pi(2e - 2f)\\ &= 2\pi(v-2)\\ &= 2\pi \cdot v - 4\pi \quad ■\end{aligned}$$

This theorem says that the sum of all angles in all the faces of a polyhedron is by 4π smaller than if the angles at every vertex would sum up to 2π. In other words, the total angular defect of any bounded convex 3-dimensional polyhedron is 4π. This is closely related to the polyhedral curvature and the Gauss-Bonnet theorem.

3.4. Regular polyhedra

The ancient Greeks were the first to consider the question of what convex polyhedra have congruent regular polygons for faces, and that an equal number of faces meet at a vertex. Such polyhedra are called regular polyhedra and already Euclid in his Elements proved that there are exactly 5 regular 3-dimensional convex polyhedra.

We will consider the related question, with somewhat relaxed condition of regularity. Namely, we will consider the question of determining the combinatorially regular polyhedra, all of whose faces have the same number p of edges and the same number q of faces meet at each vertex (without the metrical condition on the regularity of faces).

THEOREM 3.8. *There are exactly* 5 *combinatorially regular* 3*-dimensional convex polyhedra.*

Proof. Let the faces of a regular polyhedron P be p-gons and q edges share the common vertex. Since each edge belongs to 2 faces, we have $f \cdot p = 2e$, and since each edge has 2 vertices we have $v \cdot q = 2e$. By determining f and v from these equalities, and substituting in the Euler formula, we get:

$$\frac{2e}{q} - e + \frac{2e}{p} = 2.$$

By dividing by $2e$ we get:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}.$$

Since $p, q \geq 3$, we have only 5 integer solutions for this equation presented in the following table.

p	q	e	v	f	name
3	3	6	4	4	tetrahedron
4	3	12	8	6	cube
3	4	12	6	8	octahedron
3	5	30	12	20	icosahedron
5	3	30	20	12	dodecahedron

This means that there are at most 5 combinatorially regular polyhedra. But, the 5 polyhedra with the above values are realizable (even as metrically regular polyhedra, by Euclid), and so there are exactly 5 combinatorially regular polyhedra. ■

The interesting question to consider is the determination of the number of regular polyhedra in higher dimensions. Certainly, in every dimension d there are at least 3 metrically regular polyhedra, and those are: the simplex, the hyper-cube (with 2^d vertices at the points $(\pm 1, \pm 1, \ldots, \pm 1)$), and the hyper-octahedron (with $2d$ vertices at the points $e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d$). In the Euclidean space $\mathbf{R}^d$ of the dimension $d \geq 5$, there are no other regular polyhedra. However, in $\mathbf{R}^4$, there are 3 more regular polyhedra, and altogether 6. The proofs of these facts are beyond the scope of this text.

As for another application of the Euler formula, let us consider a relaxed condition of regularity, that every vertex of the convex 3-dimensional polyhedron belongs to the same number q of edges. We say that the graph of such a polyhedron is q-valent. We show a property that polyhedra whose graphs are 3-valent have to satisfy.

COROLLARY 3.9. *For a polyhedron P whose graph is 3-valent, the following equality is true (where again p_k is the number of k-gonal faces of P):*

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k\geq 7}(k-6)p_k.$$

Proof. Since each vertex belongs to 3 edges, we have $3v = 2e$. If we multiply the Euler formula by 6 and substitute $6v = 4e$ in it, we get $6f - 2e = 12$. This implies:

$$6\sum_{k\geq 3} p_k - \sum_{k\geq 3} kp_k = 12,$$

and the corollary follows. ∎

It is interesting to mention here the Eberhard's theorem which says that an inverse variant is also true. Namely, for given values of numbers of k-gonal faces for $k \geq 3$ and $k \neq 6$, satisfying the equality from the above corollary, there is a value of p_6 and the convex 3-dimensional polyhedron whose graph is 3-valent, having p_k faces which are k-gons for all $k \geq 3$.

The similar statement is true for polyhedra whose graphs are 4-valent. We formulate this as an exercise.

EXERCISE 3.10. For a polyhedron whose graph is 4-valent the following equality is true:

$$p_3 = 8 + \sum_{k\geq 5}(k-4)p_k.$$

Again, a similar inverse variant is true. Namely, for given values of p_k for $k \geq 3$ and $k \neq 4$, there is a value of p_4 and the convex 3-dimensional polyhedron whose graph is 4-valent and which has p_k faces with k vertices for every $k \geq 3$.

EXERCISE 3.11. Find the equality satisfied by polyhedra whose graphs are 5-valent. Show that no polyhedron could have k-valent graph for $k \geq 6$.

3.5. Euler-Poincaré characteristic

The Euler formula and its generalizations were the motivating force for the foundation and the development of topology, and algebraic topology in particular (see [1]).

A great part of this was done by H. Poincaré. For a convex polyhedron P of higher dimension d, he showed that the equality

$$f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^{d-1}f_{d-1} = 1 - (-1)^d$$

holds true, where f_k denotes the number of k-dimensional faces of P. The Euler formula is a special case when $d = 3$. The left hand side of the above equality is called the *Euler-Poincaré characteristic* of P and it is denoted by $\chi(P)$. Poincaré considered this characteristic for non-convex polyhedra as well, and showed that it is a topological invariant and that it does not depend on the combinatorial structure of a polyhedron.

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