THE TEACHING OF MATHEMATICS

2005, Vol. VIII, 1, pp. 15–29

INTERCHANGING TWO LIMITS

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This paper is dedicated to the memory of our illustrious professor of analysis Slobodan Aljančić (1922–1993).

Abstract. By the use of convenient metrics, the ordered set of natural numbers plus an ideal element and the partially ordered set of all partitions of an interval plus an ideal element are converted into metric spaces. Thus, the three different types of limit, arising in classical analysis, are reduced to the same model of the limit of a function at a point. Then, the theorem on interchange of iterated limits, valid under the condition that one of the iterated limits exists and the other one exists uniformly, is used to derive a long sequence of statements of that type that are commonly present in the courses of classical analysis. All apparently varied conditions accompanying such statements are, then, unmasked and reduced to one and the same: one iterated limit exists and the other one exists uniformly.

ZDM Subject Classification: I35; *AMS Subject Classification*: 00 A 35

Key words and phrases: Interchange of two limits, uniform convergence, definite integral as a limit.

1. Introduction

The simple concept of metric space proves as a convenient setting for a unification of different types of convergence and continuity that arise in classical analysis. For this reason, this concept finds its place in the contents of the first contemporary courses of mathematical analysis. Particularly, by replacing the absolute value with a distance function, immediate generalizations of conditions determining classical concepts of convergence and continuity are obtained.

Due to different terms, the following three types of limit are present in the classical analysis

$$\lim a_n, \qquad \lim f(x), \qquad \lim \sigma(f,P),$$

being respectively: the limit of a sequence, the limit of a function at a point and the limit of integral sums. To have them as particular cases of a more general concept of limit, generalized sequences (nets) or filters are used (and they are not common topics on the list of themes for the first courses in analysis).

Our idea of integrating these different types of limit is based on the procedure of introducing metric on the set $\mathbf{N}$ of natural numbers together with an ideal point and on the set of all partitions of an interval together with an ideal point. Then, the model of the limit at a point of a function mapping a metric space into another one embraces all three types of limit present in the classical analysis. Then, our main objective in this paper is to use (and prove) a theorem on the interchange of two limits under the condition that one of them exists and the other one exists uniformly. Relying on this theorem, we derive proofs of a (long) sequence of statements on interchange of two limits that are commonly present in the analysis courses.

The effects of such an approach are also seen in the fact that an apparently varied set of conditions associated with these particular statements translates uniquely into the requirement: one of the two limits exists and the other one exists uniformly.

Now we turn our attention to the history of the 19th century mathematics to recall how the concept of uniform convergence, though used explicitly, did not become a common property for a longer time. In his famous Course d'analyse, (l821), Cauchy (A-L. Cauchy, 1789–1857) made a misstep with respect to rigor, stating that the sum of a convergent series of continuous functions is a continuous function. In 1826, Abel (N. H. Abel, 1802–1829) restricted this statement to the case of uniform convergence, giving a correct proof of it (Jour. für Math. 1, 311–339), but not isolating the property of uniform convergence. Both men, Stokes (G. G. Stokes, 1819–1903) (Trans. Camb. Phil. Soc., 85, 1848) and Seidel (Ph. L. von Seidel, 1821–1896) (Abh. der Bayer. Akad. der Wiss., 1847/49) recognized the distinction between uniform and point-wise convergence and the significance of the former in the theory of infinite series. Further on, from a letter of Weierstrass (K. Weierstrass, 1815–1897) (unpublished till 1894 (Werke)) it is seen that he must have drawn this distinction as early as 1841. Nevertheless, Cauchy was the first to recognize ultimately the uniform convergence as a property assuring the continuity of the sum of a series of continuous functions (Comp. Rend., 36, 1853). It was Weierstrass who used first the concept of uniform convergence as a condition for term by term integration of a series as well as for differentiation under the integral sign. Through the circle of his students the importance of this concept was made widely known.

At the end, let us note that, when preparing this paper, we used an unpublished manuscript of the second author.

2. Iterated limits

Let A be a nonempty subset of a metric space (M_1, d_1) and B a nonempty subset of a metric space (M_2, d_2). Let $f\colon A \times B \to M$ be a mapping into a complete metric space (M, d). Denote by X' the set of accumulation points of a subset X of a metric space. Note that $A' \times B' \subset (A \times B)'$, which follows from

$$\begin{aligned}
&(x_0, y_0) \in A' \times B' \\
&\qquad \iff (\forall \varepsilon > 0)\ \{[K(x_0, \varepsilon) \setminus \{x_0\}] \cap A \neq \varnothing \text{ and } [K(y_0, \varepsilon) \setminus \{y_0\}] \cap B \neq \varnothing\} \\
&\qquad \implies (\forall \varepsilon > 0)\, [K(x_0, \varepsilon) \times K(y_0, \varepsilon) \setminus \{(x_0, y_0)\}] \cap A \times B \neq \varnothing \\
&\qquad \iff (x_0, y_0) \in (A \times B)'.
\end{aligned}$$

Let $x_0 \in A'$ and $y_0 \in B'$. A limit $\lim_{y \to y_0} f(x,y) = \varphi(x)$ exists for each $x \in A$ if the following condition holds:

$$(\forall x \in A)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B)(0 < d_2(y, y_0) < \delta \implies d(f(x,y), \varphi(x)) < \varepsilon);$$

and one can formulate similarly the condition for the existence of $\lim_{x \to x_0} f(x,y) = \psi(y)$. Let us strengthen this condition demanding that the number δ does not depend on x. Practically, this means that the quantifier $(\forall x \in A)$ has to be placed after the quantifier $(\exists \delta > 0)$. Thus we obtain that $\lim_{y \to y_0} f(x,y) = \varphi(x)$ exists *uniformly in* $x \in A$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)(\forall y \in B)(0 < d_2(y, y_0) < \delta \implies d(f(x,y), \varphi(x)) < \varepsilon);$$

and the condition for the existence of $\lim_{x \to x_0} f(x,y) = \psi(y)$, uniformly in $y \in B$ is formulated similarly.

Cauchy condition for the existence of the $\lim_{y \to y_0} f(x,y) = \varphi(x)$, uniformly in $x \in A$ is

$$\begin{gathered}(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)(\forall y' \in B)(\forall y'' \in B)\\ 0 < d_2(y', y_0) < \delta \text{ and } 0 < \delta_2(y'', y_0) < \delta \implies d(f(x,y'), f(x,y'')) < \varepsilon.\end{gathered}$$

Since the space (M, d) is assumed to be complete, it is clear that this condition is necessary and sufficient for the existence of $\lim_{y \to y_0} f(x,y) = \varphi(x)$, uniformly in $x \in A$.

Writing formally,

$$\begin{aligned}\lim_{x \to x_0} \varphi(x) &= \lim_{x \to x_0} (\lim_{y \to y_0} f(x,y)),\\ \lim_{y \to y_0} \psi(x) &= \lim_{y \to y_0} (\lim_{x \to x_0} f(x,y)),\end{aligned}$$

we call these expressions *iterated limits* (which, of course need not exist). In the sequel, we shall omit additional parentheses. In this context, the expression

$$\lim_{(x_0, y_0)} f(x,y)$$

is called *double limit* (and it may equally be non-existent).

A general connection between the mentioned limits is given by the following

Proposition 1. *Let $f\colon A \times B \to M$ be a function from a subset $A \times B \subset M_1 \times M_2$ into M and $(x_0, y_0) \in A' \times B'$, where M_1, M_2 and M are metric spaces. If*

(i) $\lim_{(x_0, y_0)} f(x,y) = \alpha$ *exists,*

(ii) *for each $y \in B$, $\lim_{x \to x_0} f(x,y) = \psi(y)$ exists,*

then $\lim_{y \to y_0} \psi(y)$ exists and is equal to α.

Proof. Condition (i) implies that

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall(x,y)\in A\times B)$$
$$(x,y)\in K(x_0,\delta)\times K(y_0,\delta)\setminus\{(x_0,y_0)\}\implies d(f(x,y),\alpha)<\frac{\varepsilon}{2}.$$

According to (ii), using the continuity of function d, there exists

$$\lim_{x\to x_0} d(f(x,y),\alpha)=d(\psi(y),\alpha)\leqslant\frac{\varepsilon}{2}.$$

Thus,

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall y\in B)\ y\in K(y_0,\delta)\setminus\{y_0\}\implies d(\psi(y),\alpha)<\varepsilon,$$

which means that $\lim\limits_{y\to y_0}\psi(y)=\alpha$. ■

Note that the conditions of the previous proposition are pretty strong, especially this is the case with condition (i). Therefore, this proposition is of small practical value. However, it can be used to prove the non-existence of some double limits.

Example 1. Consider the following three functions:

$$f(x,y)=\frac{x-y+x^2+y^2}{x+y},\qquad g(x,y)=\frac{x\sin\frac{1}{x}+y}{x+y},\qquad h(x,y)=x\sin\frac{1}{y},$$

all three defined on the set $A\times B=(0,+\infty)\times(0,+\infty)$.

For the function f, $\lim\limits_{y\to0}f(x,y)=1+x$, for each $x\in A$ and $\lim\limits_{x\to0}f(x,y)=-1+y$, for each $y\in B$, and so

$$\lim_{x\to0}\lim_{y\to0}f(x,y)=1\neq-1=\lim_{y\to0}\lim_{x\to0}f(x,y).$$

The double limit does not exist.

For the function g, $\lim\limits_{y\to0}g(x,y)=\sin\dfrac{1}{x}$, for each $x\in A$ and $\lim\limits_{x\to0}g(x,y)=1$, for each $y\in B$. Hence, $\lim\limits_{y\to0}\lim\limits_{x\to0}g(x,y)=1$, and both $\lim\limits_{x\to0}\lim\limits_{y\to0}g(x,y)$ and the double limit do not exist.

For the function h, using that $0\leqslant\left|x\sin\dfrac{1}{y}\right|\leqslant|x|$, we obtain that $\lim\limits_{(0,0)}h(x,y)=0$. On the other hand, $\lim\limits_{y\to0}h(x,y)$ does not exist for any $x\in A$, and so the respective iterated limit does not exist either. The other iterated limit exists and $\lim\limits_{y\to0}\lim\limits_{x\to0}h(x,y)=0$. △

Now we shall modify the conditions of Proposition 1, demanding that one of the limits $\lim\limits_{x\to x_0}f(x,y)$ and $\lim\limits_{y\to y_0}f(x,y)$ exists, and that the other one exists uniformly. We shall conclude that the three limits, both iterated and the double one, all exist and are equal. Such a statement synthesizes then a series of classical results on interchange of limit operations. In terms of generalized sequences it is called Moore Theorem (E. H. Moore, 1891–1976)[1].

THEOREM 1. (The theorem on interchange of two limits) *Let $f\colon A \times B \to M$ be a mapping into a complete metric space, where A and B are subsets of metric spaces M_1 and M_2, respectively, and let $x_0 \in A' \setminus A$, $y_0 \in B' \setminus B$. If*

(i) $\lim\limits_{x \to x_0} f(x, y) = \psi(y)$ *exists for each* $y \in B$;

(ii) $\lim\limits_{y \to y_0} f(x, y) = \varphi(x)$ *exists uniformly in* $x \in A$,

then the three limits

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y), \qquad \lim_{y \to y_0} \lim_{x \to x_0} f(x, y), \qquad \textit{and} \qquad \lim_{(x_0, y_0)} f(x, y)$$

all exist and are equal.

Proof. Let $\varepsilon > 0$ be arbitrary. According to condition (ii), we have

$$(1) \qquad (\exists \delta > 0)(\forall y \in B)\ 0 < d_2(y, y_0) < \delta \implies (\forall x \in A)\, d(f(x, y), \varphi(x)) < \frac{\varepsilon}{6}.$$

Let $\overline{y} \in K(y_0, \delta) \setminus \{y_0\}$. Using condition (i) we obtain

$$(2) \qquad (\exists \overline{\delta} > 0)(\forall x \in A)\ 0 < d_1(x, x_0) < \overline{\delta} \implies d(f(x, \overline{y}), \psi(\overline{y})) < \frac{\varepsilon}{6}.$$

Let us take a neighborhood of the point (x_0, y_0) of the form

$$U = K(x_0, \overline{\delta}) \times K(y_0, \delta)$$

and let points (x', y'), (x'', y'') belong to the set $U \setminus \{(x_0, y_0)\}$. Using the triangle inequality, we get

$$\begin{aligned} d(f(x', y'), f(x'', y'')) \leqslant{} & d(f(x', y'), \varphi(x')) + d(\varphi(x'), f(x', \overline{y})) \\ & + d(f(x', \overline{y}), \psi(\overline{y})) + d(\psi(\overline{y}), f(x'', \overline{y})) \\ & + d(f(x'', \overline{y}), \varphi(x'')) + d(\varphi(x''), f(x'', y'')). \end{aligned}$$

By (2), the third and the fourth summand on the right-hand side are both less than $\varepsilon/6$, and by (1), each of the other four summands is less than $\varepsilon/6$. Thus,

$$\begin{aligned} &(\forall (x', y') \in A \times B)(\forall (x'', y'') \in A \times B) \\ &\qquad (x', y'), (x'', y'') \in U \setminus \{(x_0, y_0)\} \implies d(f(x', y'), f(x'', y'')) < \varepsilon, \end{aligned}$$

and so the function f satisfies Cauchy condition at the point (x_0, y_0). Therefore, since the space M is complete, there exists

$$\lim_{(x_0, y_0)} f(x, y) = \alpha.$$

[1]See N. Dunford and J. T. Schwartz, *Linear Operators, Part I, Interscience Publishers*, New York, 1958.

Using Proposition 1 and condition (i), we have

$$\lim_{y\to y_0} \psi(y) = \alpha = \lim_{y\to y_0} \lim_{x\to x_0} f(x,y),$$

and, by (ii),

$$\lim_{x\to x_0} \varphi(x) = \alpha = \lim_{x\to x_0} \lim_{y\to y_0} f(x,y). \quad \blacksquare$$

We shall apply the obtained result to a sequence (f_n) of functions with domain A which is a subset of a metric space M_0 and codomain which is a metric space M. Then, we can consider $f_n(x)$ as a function of two variables, taking

$$\varphi(n,x) = f_n(x).$$

Since $\mathbf{N}$ can be understood as a subset of the metric space $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ with the metric

$$d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|, \qquad \left(\frac{1}{\infty} = 0\right)$$

and since $\infty \in \mathbf{N}'$, for the function $\varphi\colon \mathbf{N}\times A \to M$, the fact that for each $x \in A$, $\lim_{n\to\infty} \varphi(n,x) = f(x)$ exists, simply means that the sequence (f_n) converges to the function f on the set A.

The fact that $\lim_{n\to\infty} \varphi(n,x) = f(x)$ *exists uniformly in* $x \in A$, *is equivalent to the fact that the sequence* (f_n) *converges uniformly on* A. The respective condition

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall n \in \mathbf{N})(\forall x \in A)\ 0 < d(n,\infty) < \delta \implies d(\varphi(n,x), f(x)) < \varepsilon,$$

can be written as

$$(\forall \varepsilon > 0)(\exists m \in \mathbf{N})(\forall n \in \mathbf{N})(\forall x \in A)\ n > m \implies d(f_n(x), f(x)) < \varepsilon,$$

taking that $d(n,\infty) = \dfrac{1}{n}$ and $\left[\dfrac{1}{\delta}\right] = m$.

As an example of using such treatment of functional sequences as functions of two variables, we shall derive now the well-known theorem about continuity of the limit function.

Corollary 1. *Let* (f_n) *be a uniformly convergent sequence of continuous functions from a metric space* M_0 *into a complete metric space* M. *Then the limit function* $f = \lim f_n$ *is also continuous.*

Proof. If $x_0 \in M_0$ is an isolated point, the assertion is trivial. Let $x_0 \in M_0'$. For the function

$$\varphi(n,x) = f_n(x), \qquad (M_0 \subset M_0, \quad \mathbf{N} \subset \mathbf{N}^*)$$

we have also

(i) $\lim_{n\to\infty} \varphi(n,x)$ exists uniformly in $x \in M_0$,

and by the continuity of all the functions f_n at the point x_0,

(ii) for each $n \in \mathbf{N}$, $\lim\limits_{x\to x_0} \varphi(n,x) = f_n(x_0)$ exists,

and by Theorem 1, it follows that

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0}\lim_{n\to\infty} \varphi(n,x) = \lim_{n\to\infty}\lim_{x\to x_0} \varphi(n,x)$$
$$= \lim_{n\to\infty} f_n(x_0) = f(x_0). \quad \blacksquare$$

In a similar way, the following assertion on termwise differentiation can be proved.

COROLLARY 2. *Let (f_n) be a sequence of real-valued functions, converging on $[a,b]$ to a function f, and let*

(i) *f_n be a differentiable function for each $n \in \mathbf{N}$,*

(ii) *the sequence (f_n') converges uniformly on $[a,b]$.*

Then the function f is differentiable and the equality

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

holds.

The respective properties of functional series can be derived now as easy consequences.

3. Double sequences

A function $a\colon \mathbf{N}\times\mathbf{N} \to \mathbf{R}$ is called a *double sequence*. As usual, we shall denote the value of the function a at (i,j) as a_{ij}, and this sequence will be denoted by (a_{ij}). The following theorem gives sufficient conditions for the interchange of the order of summation.

THEOREM 2. *Let (a_{ij}) be a double sequence such that:*

(i) $(\forall i)$ $\sum\limits_{j=1}^{\infty} |a_{ij}| = b_i < +\infty$,

(ii) $\sum\limits_{i=1}^{\infty} b_i$ *is a convergent series.*

Then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof. Let $f_i\colon \mathbf{N}^* = \mathbf{N}\cup\{\infty\} \to \mathbf{R}$ be the function defined by

$$f_i(k) = \sum_{j=1}^{k} a_{ij}.$$

For each i, the function f_i is continuous on $\mathbf{N}^*$, because

$$\lim_{k\to\infty} f_i(k) = \sum_{j=1}^{\infty} a_{ij} = f_i(\infty).$$

The series $\sum_{i=1}^{\infty} f_i(k)$ converges uniformly according to the Weierstrass Test, because

$$\text{(i)}\ \ (\forall i)|f_i(k)| \leqslant b_i \quad \text{and} \quad \text{(ii)}\ \ \sum_{i=1}^{\infty} b_i \text{ converges.}$$

Therefore,

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} f_i(k) = \sum_{i=1}^{\infty} \lim_{k\to\infty} f_i(k).$$

The desired equality follows from

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} f_i(k) = \lim_{k\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{k} a_{ij} = \lim_{k\to\infty} \sum_{j=1}^{k} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij},$$

$$\sum_{i=1}^{\infty} \lim_{k\to\infty} f_i(k) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}. \quad \blacksquare$$

4. The Toeplitz Limit Theorem

The following theorem was proved by Toeplitz (O. Toeplitz, 1881–1940) in Prace mat.-fiz. 22 (1911), 113–119.

THEOREM 3. *The coefficients of the matrix*

$$\begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} & \dots \\ a_{10} & a_{11} & \dots & a_{1n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{k0} & a_{k1} & \dots & a_{kn} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

are assumed to satisfy the following two conditions:

(a) *for each fixed n, $a_{kn} \to 0$ as $k \to \infty$,*

(b) *there exists a constant K such that for each fixed k and any n,*

$$|a_{k0}| + |a_{k1}| + \cdots + |a_{kn}| < K.$$

Then, for every null sequence $(z_0, z_1, \dots, z_n, \dots)$, the numbers

$$z'_k = a_{k0}z_0 + a_{k1}z_1 + \cdots + a_{kn}z_n + \cdots$$

also form a null sequence.

Proof. Denote

$$\varphi(n,k) = a_{k0}z_0 + a_{k1}z_1 + \cdots + a_{kn}z_n.$$

The assumption (a) implies that

(a') $\lim_k \varphi(n,k) = 0$, for each n.

On the other hand,

(b') $\lim_n \varphi(n,k) = \sum_{j=0}^{\infty} a_{kj} z_j = \psi(k)$ exists, uniformly in k.

To prove it, observe first that, by (b), the series $\sum_{j=0}^{\infty} a_{kj}$ converges absolutely. Fix $\varepsilon > 0$. Let j_0 be such that

$$j > j_0 \implies |z_j| < \frac{\varepsilon}{K}.$$

Then, for $j > j_0$, we have

$$|\psi(k) - \varphi(n,k)| = \left| \sum_{j=n+1}^{\infty} a_{kj} z_j \right| \leqslant \sum_{j=n+1}^{\infty} |a_{kj}| \cdot \max_j |z_j| < K \cdot \frac{\varepsilon}{K} = \varepsilon,$$

and so $\lim_n \varphi(n,k)$ exists uniformly in k.

Now, by Theorem 1, we obtain that

$$\lim_k z'_k = \lim_k \lim_n \varphi(n,k) = \lim_n \lim_k \varphi(n,k) = 0. \quad \blacksquare$$

COROLLARY 3. *If, in addition to assumption (a) of Theorem 3,*

$$(\forall k) \;\; a_{k0} + a_{k1} + \cdots + a_{kn} \to 1 \quad as \quad n \to \infty,$$

and $z_n \to z_0$*, then* $z'_k \to z_0$ *as* $k \to \infty$.

5. Integral as a limit

Denote by Π the set of all partitions P of a segment $[a,b]$. For the partition P given by $a = x_0 < x_1 < \cdots < x_n = b$, the number

$$\|P\| = \max\{\, x_i - x_{i-1} \mid i = 1, 2, \ldots, n \,\}$$

is called the *norm of partition* P. Let ∞ be an element which does not belong to Π and let $\|\infty\| = 0$. The set $\Pi^* = \Pi \cup \{\infty\}$, together with the metric

$$d_\Pi(P_1, P_2) = \begin{cases} \|P_1\| + \|P_2\|, & P_1 \neq P_2, \\ 0, & P_1 = P_2, \end{cases}$$

is a metric space which will be called the *partition space* and will be denoted by (Π^*, d_Π). If, for two partitions P_1 and P_2, $\|P_1\| \leqslant \|P_2\|$, we say that P_1 is smaller than P_2. Note that a finer partition is also smaller, but the converse is not true.

For a partition P, let $\mathring{P}$ be the partition which has as partitioning points all the points of partition P and, moreover, all the midpoints of segments of P. Then $\|\mathring{P}\| = \frac{1}{2}\|P\|$.

Let P_0 be the partition with $a = x_0 < x_1 = b$. Let us define a sequence (P_n) of partitions, taking $P_{n+1} = \mathring{P}_n$. Then $\|P_n\| = \dfrac{b-a}{2^n}$, and so

$$\lim_{n\to\infty} d_\Pi(\infty, P_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0,$$

wherefrom it follows that ∞ is an accumulation point of the set Π in the space Π^*.

Let $f\colon [a,b] \to \mathbf{R}$ be a bounded function, $s(f,P)$ and $S(f,P)$ be its lower and its upper sum, with respect to the partition P. These sums can be considered as functions

$$s\colon \Pi \to \mathbf{R}, \qquad S\colon \Pi \to \mathbf{R}$$

from the space of partitions into $\mathbf{R}$. The limit $\lim\limits_{\infty} s(f,P) = \alpha$ exists if the condition

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \Pi)\ 0 < d_\Pi(P,\infty) < \delta \implies |\alpha - s(f,P)| < \varepsilon$$

holds, or, equivalently, if the condition

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \Pi)\ \|P\| < \delta \implies |\alpha - s(f,P)| < \varepsilon$$

holds. The following proposition associates this limit with the lower (upper) integral of the function f.

Proposition 2. *Let $f\colon [a,b] \to \mathbf{R}$ be a bounded function. Then $\lim\limits_{\infty} s(f,P)$ (resp. $\lim\limits_{\infty} S(f,P)$) exists and is equal to $\sup_P s(f,P)$ (resp. $\inf_P S(f,P)$).*

Proof. Let $\sup_P s(f,P) = \underline{I}$. Since the function f is bounded, there exists a constant $M > 0$, such that for each $x \in [a,b]$, $|f(x)| \leqslant M$. Let $\varepsilon > 0$. There exists a partition P_1 with partitioning points $a = x_0 < x_1 < \dots < x_k = b$, such that

$$\underline{I} - s(f,P_1) < \frac{\varepsilon}{2}.$$

Let $\delta = \dfrac{\varepsilon}{8Mk}$. Let $P \in \Pi$ be such that $\|P\| < \delta$. Denote by $\overline{P}$ the superposition of partitions P and P_1. We have now

$$s(f,\overline{P}) - s(f,P) = \sum_{\Delta \in \overline{P}} m_\Delta v(\Delta) - \sum_{\Delta \in P} m_\Delta v(\Delta),$$

where $m_\Delta = \inf\{\, f(x) \mid x \in \Delta \,\}$ and $v(\Delta)$ is the length of segment Δ.

Since partition $\overline{P}$ is finer than partition P, for each $\Delta \in \overline{P}$ there exists a unique $\Delta' \in P$ such that $\Delta \subset \Delta'$. Let $m'_\Delta = m_{\Delta'}$, and so

$$\sum_{\Delta \in P} m_\Delta v(\Delta) = \sum_{\Delta \in \overline{P}} m'_\Delta v(\Delta).$$

Now,

$$s(f,\overline{P}) - s(f,P) = \sum_{\Delta \in \overline{P}} (m_\Delta - m'_\Delta) v(\Delta).$$

We have $m_\Delta - m'_\Delta = 0$ when Δ has no endpoints belonging to partition P_1, except points a and b, and $m_\Delta - m'_\Delta \neq 0$ when an endpoint of segment Δ belongs to the set $\{x_1, \dots, x_{k-1}\}$, and this endpoint does not belong to P. The number of such segments is at most $2(k-1)$. Since $|m_\Delta - m'_\Delta| \leqslant 2M$, we have

$$s(f,\overline{P}) - s(f,P) \leqslant 2(k-1) \cdot 2M \cdot \|\overline{P}\| < 2(k-1) \cdot 2M \cdot \frac{\varepsilon}{8Mk} = \frac{\varepsilon}{2}.$$

Hence, when $\|P\| < \delta$, we have

$$\underline{I} - s(f,P) \leqslant (\underline{I} - s(f,\overline{P}) + (s(f,\overline{P}) - s(f,P)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

because $\overline{P}$ is also finer that P_1 and so

$$\underline{I} - s(f,\overline{P}) \leqslant \underline{I} - s(f,P_1) < \frac{\varepsilon}{2}.$$

This completes the proof that $\lim\limits_{\infty} s(f,P) = \underline{I}$.

The equality $\lim\limits_{\infty} S(f,P) = \overline{I} = \inf_P S(f,P)$ can be proved in a similar way. ∎

If the function f is integrable, then

$$\lim_{\infty} s(f,P) = \lim_{\infty} S(f,P) = \int_a^b f(x)\,dx.$$

For a partition P, let ξ_P be a choice function, i.e., let $\xi_P(\Delta) \in \Delta$. The sum

$$\sigma(f,P) = \sum \{\, f(\xi_P(\Delta))v(\Delta) \mid \Delta \in P \,\}$$

is called the *integral sum of the function f, corresponding to the partition P*. Since

$$s(f,P) \leqslant \sigma(f,P) \leqslant S(f,P),$$

for each ξ_P, we have

$$\lim_{\infty} \sigma(f,P) = \int_a^b f(x)\,dx,$$

for each integrable function f.

We shall consider now the question of interchanging a limit and an integral. First, we need an auxiliary statement.

Lemma 1. *Let $f\colon [a,b] \to \mathbf{R}$ and $g\colon [a,b] \to \mathbf{R}$ be two functions, such that*

$$(\forall x \in [a,b])\ |f(x) - g(x)| < k.$$

Then, for each nonempty set $S \subset [a,b]$, the inequalities

$$|\inf_{x\in S} f(x) - \inf_{x\in S} g(x)| < 2k, \qquad |\sup_{x\in S} f(x) - \sup_{x\in S} g(x)| < 2k.$$

hold true.

Proof. Let $\underline{x} \in S$ be a point, such that $f(\underline{x}) < m_S(f) + k$, where $m_S(f) = \inf\limits_{x\in S} f(x)$. Then,

$$m_S(f) > f(\underline{x}) - k > g(\underline{x}) - 2k \geqslant m_S(g) - 2k,$$

where $m_S(g) = \inf\limits_{x\in S} g(x)$. Similarly,

$$m_S(g) > m_S(f) - 2k,$$

which proves that $|m_S(f) - m_S(g)| < 2k$.

The proof of the other part of the proposition is similar. ∎

THEOREM 4. *Let (f_n) be a sequence of functions, integrable on the segment $[a,b]$, and let (f_n) converges uniformly to a function f. Then, the function f is integrable and the equality*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx=\int_a^b f(x)\,dx$$

is valid.

Proof. Let $\varepsilon>0$ be given. Since the sequence (f_n) converges uniformly to the function f, there exists an $m\in\mathbf{N}$ such that

$$(\forall n\in\mathbf{N})(\forall x\in[a,b])\; n\geqslant m \implies |f_n(x)-f(x)|<\frac{\varepsilon}{2(b-a)}.$$

The lower sum of the function f_n for the partition P,

$$s(f_n,P)=\textstyle\sum\{\, m_n(\Delta)v(\Delta)\mid \Delta\in P\,\},$$

is a function from $\mathbf{N}\times\Pi\subset\mathbf{N}^*\times\Pi^*$ into $\mathbf{R}$. For $n\geqslant m$ and each P, we have

$$|s(f.P)-s(f_n,P)|\leqslant\sum_{\Delta\in P}|m_n(\Delta)-m(\Delta)|v(\Delta).$$

Applying Lemma 1, we obtain

$$|s(f,P)-s(f_n,P)|<2\cdot\frac{\varepsilon}{2(b-a)}(b-a)=\varepsilon,$$

and so $\lim\limits_{n\to\infty}s(f_n,P)=s(f,P)$ exists uniformly in P. For each $n\in\mathbf{N}$,

$$\lim_{P\to\infty}s(f_n,P)=\int_a^b f_n(x)\,dx$$

exists, because f_n is an integrable function. Applying Theorem 1, it follows that both iterated limits exist and are equal. Hence, we have

$$\lim_P\lim_n s(f_n,P)=\lim_P s(f,P)=\underline{\int_a^b} f(x)\,dx,$$

$$\lim_n\lim_P s(f_n,P)=\lim_n\int_a^b f_n(x)\,dx,$$

and

$$\underline{\int_a^b} f(x)\,dx=\lim_n\int_a^b f_n(x)\,dx,$$

Repeating the previous arguments and using upper sums, the equality

$$\overline{\int_a^b} f(x)\,dx=\lim_n\int_a^b f_n(x)\,dx$$

can also be proved. ■

Note that the previous theorem remains valid when the domain of the given functions is an arbitrary set $A \subset \mathbf{R}$, which is Jordan measurable (C. M. E. Jordan, 1838–1922). Namely, such a set A is bounded and there exists a segment $[a, b]$ such that $A \subset [a, b]$. If the functions $f_n \colon A \to \mathbf{R}$ are integrable, so are the functions

$$\overline{f}_n \colon [a, b] \to \mathbf{R} \quad \text{given by} \quad \overline{f}_n(x) = \begin{cases} f_n(x), & x \in A, \\ 0, & x \in [a, b] \setminus A. \end{cases}$$

If the sequence (f_n) converges uniformly to the function $f \colon A \to \mathbf{R}$, then the sequence $(\overline{f}_n)$ converges uniformly to the function

$$\overline{f}(x) = \begin{cases} f(x), & x \in A, \\ 0, & x \in [a, b] \setminus A, \end{cases}$$

which is also integrable. According to the previous theorem, we have

$$\lim_{n\to\infty} \int_A f_n(x)\, dx = \lim_{n\to\infty} \int_a^b \overline{f}_n(x)\, dx = \int_a^b \overline{f}(x)\, dx = \int_A f(x)\, dx.$$

Assuming that the limit function of a convergent sequence (f_n) is integrable, the assumption of uniform convergence can be weakened. First, we formulate and prove an auxiliary statement, which is of interest on its own. Let (f_n) be a sequence of real functions, converging to a function f on a set $A \subset \mathbf{R}$. Let $\overline{A}$ be the closure of the set A in $\mathbf{R}^*$. We shall say that (f_n) *converges uniformly around a point* $x_0 \in \overline{A}$ if there exists an open neighborhood U of the point x_0, such that (f_n) converges uniformly on the set $U \cap A$.

PROPOSITION 3. *Let f_n, $n \in \mathbf{N}$, be real functions with domain $A \subset \mathbf{R}$. The sequence (f_n) converges uniformly in $x \in A$ if and only if it converges uniformly around each point of the set $\overline{A} \subset \mathbf{R}^*$.*

Proof. If the sequence (f_n) converges uniformly in $x \in A$, then it obviously converges uniformly around each point of the set $\overline{A}$.

Conversely, suppose that (f_n) converges uniformly around each point of $\overline{A}$. Then (f_n) is certainly point-wise convergent to a function f. If $\overline{A}$ contains one of the elements $-\infty$, $+\infty$ (or both of them), choose a respective neighborhood $U_{-\infty}$ or $U_{+\infty}$ (or both of them), such that (f_n) converges uniformly on $A \cap U_{-\infty}$, resp. $A \cap U_{+\infty}$. For any other point $x \in \overline{A}$, let U_x be a neighborhood such that (f_n) converges uniformly on $U_x \cap A$. The set $\overline{A} \setminus (U_{-\infty} \cup U_{+\infty})$ (if $-\infty$ or $+\infty$ does not belong to $\overline{A}$, then $U_{-\infty}$, resp. $U_{+\infty}$ is an empty set) will be compact. Hence, there exist finitely many neighborhoods

$$U_{x_1}, \quad U_{x_2}, \quad \ldots, \quad U_{x_k}$$

covering the set $\overline{A} \setminus (U_{-\infty} \cup U_{+\infty})$. Adding $U_{-\infty}$ and/or $U_{+\infty}$ to them, we obtain finitely many subsets of A, such that (f_n) converges uniformly on them. Since each point of the set A belongs to one of the sets $U_\xi \cap A$, $\xi \in \{x_1, x_2, \ldots, x_k, -\infty, +\infty\}$, the sequence (f_n) converges uniformly on A. ∎

Roughly speaking, more points are there around which the sequence (f_n) is not uniformly convergent, less regular is the convergence of (f_n). On the other hand, Proposition 3 can be formulated as follows: The sequence (f_n) does not converge uniformly in $x \in A$ if and only if it does not converge uniformly around a point $x \in \overline{A}$. Hence, nonuniform convergence can always be spotted locally.

The following is the *Lebesgue theorem* (H. L. Lebesgue, 1875–1941) *on bounded convergence for the Riemann integral* (G. F. B. Riemann, 1826–1866).

THEOREM 5. *Let (f_n) be a sequence of functions, integrable on $[a,b]$, converging to a function f. If*

(i) *the function f is integrable,*

(ii) $(\exists M \in \mathbf{R})(\forall n \in \mathbf{N})(\forall x \in [a,b])\ |f_n(x)| \leqslant M$,

(iii) *the set*

$$A = \{\, x \in [a,b] \mid (f_n) \text{ does not converge uniformly around } x \,\}$$

is a closed set of Lebesgue measure 0,

then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. Let $\varepsilon > 0$ be given. According to (iii), there exists a sequence $(\Delta_i)_{i\in\mathbf{N}}$ of open intervals covering the set A, such that

$$\sum\{\, v(\Delta_i) \mid i \in \mathbf{N} \,\} < \frac{\varepsilon}{4M}.$$

The set A is closed and bounded, and so, by Borel-Lebesgue Theorem (F. E. J. E. Borel, 1871–1956), there exist finitely many intervals $\Delta_{i_1}, \dots, \Delta_{i_k}$ which cover A, as well. Let

$$D = \Delta_{i_1} \cup \dots \cup \Delta_{i_k}.$$

The set D is Jordan measurable and

$$m(D) \leqslant v(\Delta_{i_1}) + \dots + v(\Delta_{i_k}) < \frac{\varepsilon}{4M}.$$

The sequence (f_n) converges uniformly around each point of the set $[a,b] \setminus D$, and so, by Proposition 3, it converges uniformly on $[a,b] \setminus D$. Hence, there exists an $n_0 \in \mathbf{N}$ such that

$$(\forall n \in \mathbf{N})(\forall x \in [a,b] \setminus D)\ n \geqslant n_0 \implies |f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}.$$

Now, for $n \geqslant n_0$, we have

$$\begin{aligned}\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| &\leqslant \int_D |f(x) - f_n(x)|\,dx + \int_{[a,b]\setminus D} |f(x) - f_n(x)|\,dx \\ &< 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \varepsilon,\end{aligned}$$

which proves the theorem. ■

Example 2. The sequence of functions

$$f_n(x) = \frac{nx}{1+(nx)^2}, \qquad x \in [0,1],$$

converges to the function $0\colon [0,1] \to \mathbf{R}$, and it does not converge uniformly only around the point 0. The set $A = \{0\}$ is closed and of measure 0, while

$$(\forall n \in \mathbf{N})(\forall x \in [a,b]) \ \left|\frac{nx}{1+(nx)^2}\right| \leqslant \frac{1}{2}.$$

Hence, we have

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1+(nx)^2}\,dx = 0. \quad \triangle$$

At the end, let us remark that we have not included several other cases of interchange of two limits: cases of differentiation and integration of integrals depending on parameter, equality $f''_{xy} = f''_{yx}$, etc., which can be treated in the same way.

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