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POSSIBILITIES AND LIMITATIONS OF SCIENTIFIC WORKPLACE IN STUDYING TRIGONOMETRIC FUNCTIONS

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Abstract. This paper presents the investigations regarding the role of computer in examining trigonometric functions, in particular in solving trigonometric equations and inequalities. The advantages and disadvantages of the applied procedure, are particularly emphasized.

ZDM Subject Classification: D44, G44, I44, U74; *AMS Subject Classification*: 00 A 35

Key words and phrases: CAS, teaching-learning process, trigonometry.

1. Introduction

The trigonometry is considered as the most difficult part in high school education. Therefore the examining and drawing trigonometric functions, as the application of the derivative, mainly, is not treated in the standard fourth grade teaching. This year the authors decided to work on the mentioned problems with the students talented for mathematics by using the programme package Scientific Workplace.

The teacher and the students worked together in the computer laboratory. The teacher had a computer and video beam and each two students had one computer. The programme package Scientific Workplace 3 was used by teacher and students. This programme package is quite suitable for teaching because its use is very simple, and working in this programme looks like writing on the blackboard, but the obtained objects look better. This was not the first time for students to use these programme packages. The teachers already used computer for the visualization of calculus, in particular limits, continuity, derivatives and its application, in particular for examining rational functions.

The trigonometric functions

$$\sin^3 x - \cos^3 x; \qquad \sin x - \cos x; \qquad \sin x \sin 3x$$

are considered. These functions can be found in almost all books regarding to examples in Analysis I (see [2]).

The teachers gave the analytical expression of each trigonometric function, as usual, but after that the graph of the function was drawn. This means that the procedure of solving such problems was changed. It looks like that after obtaining the graph, the problem is solved, and there is nothing else to be done. But, we still have to examine the function considering each point separately and using computer everywhere it could be used. The analysis of the functions is treated interactively, meaning that each result obtained classically is compared with the corresponding result obtained by computer. The computer provided help graphic and symbolic, meaning that the all boring calculations were done by computer.

Applying the presented procedure, with the use of computer, the new problems came out. Namely, in some examples the computer did not express all possible, or any results. We pointed out such places and, by using the usual classical explanation, we presented each part of the exact solution. The best help of computer is in enabling the excellent graphs which were used in our investigation.

The idea for mentioned considerations came from papers [8], and [9]. In the papers [1], [3] Computer Algebra System (CAS) is applied in Calculus and Analysis. In papers [4], and [5], the programme package Scientific Workplace was used as a help for teaching continuity and limits, respectively. In [6], the results of the test on continuity was analyzed. The aim of the test in [6], was to check the students knowledge of continuity after the visual presentation exposed [5]. In the paper [7] the help of programme package Scientific Workplace for drawing the graph of the rational function was presented.

2. The function $\sin^3 x - \cos^3 x$

First, we should consider the function $f(x) = \sin^3 x - \cos^3 x$, *and draw its graph, as on Figure 1, for* $-5 \le x \le 5$, *and on Figure 2, for* $-10 \le x \le 10$.

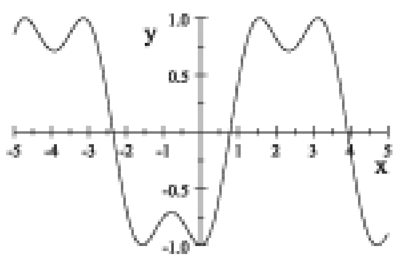

Fig. 1

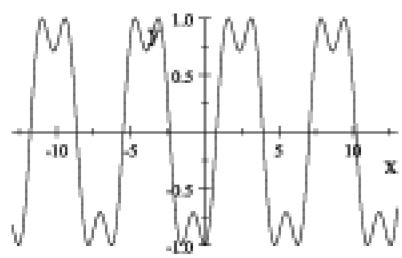

Fig. 2

1. The function is defined for all real values, and it is periodic with the main period 2π. *Figure 1 is drawn by defaults, Figure 2 is drawn in order to make clear that the main period is* 2π, *but also, the periods are:* 4π, -2π, *and* -4π. *On Figure 3, we took* $0 \le x \le 2\pi$, *as usual.*
2. This function is neither even nor odd (f is the sum of an odd function and an even one).
3. *The zeroes of f are obtained by using software as follows:*

$$f(x) = 0, \quad \textit{Solutions:} \quad \left\{x = \tfrac{1}{4}\pi\right\}, \left\{x = \arctan\left(-\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3}\right)\right\},$$
$$\left\{x = \arctan\left(-\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3}\right)\right\}.$$

Let us remark that in these cases the first zero $x = \frac{1}{4}\pi$ is determined correctly, the zero $x = \frac{5}{4}\pi$ is not determined (by software), but the points: $x = \arctan\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)$ and $x = \arctan\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)$ do not exist.

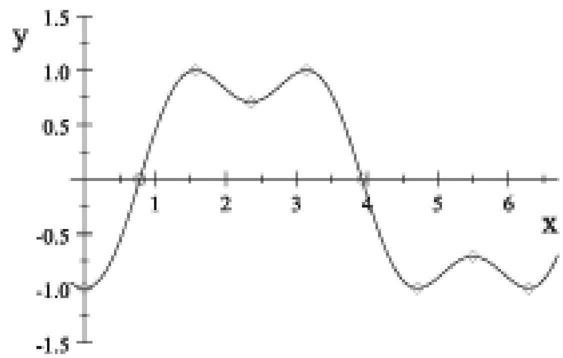

Fig. 3

Classically, the zeroes can be determined from the expression:

$$f(x) = \sin^3 x - \cos^3 x = (\sin x - \cos x)\left(1 + \frac{\sin 2x}{2}\right). \tag{1}$$

The zero of the function f is attained for such x where $\sin x = \cos x$.

From equation (1) $-1 \le \sin x \le 1$, it follows that $1 + \frac{\sin 2x}{2} > 0$, and the equation $1 + \frac{\sin 2x}{2} = 0$ has no zeroes. *Therefore, the software expressed that* $x = \arctan\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)$ *and* $x = \arctan\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)$ *are zeroes.*

In this case the students have to be very careful in their analysis, and they have to do a lot of work in determining the zeroes of functions. But it is not boring because they can be lead by the graph to the correct answer.

4. The sign of function f is easy to be determined from the form given in (1):
   $f(x) > 0, \quad x \in (\frac{\pi}{4}, \frac{5\pi}{4})$, where $\sin x > \cos x$;
   $f(x) < 0, \quad x \in (0, \frac{\pi}{4})$, $x \in (\frac{5\pi}{4}, 0)$, where $\sin x < \cos x$,
   and it can be checked, by looking at the graph, but the software cannot solve $f(x) > 0$. In fact, by using software we get:
   $$f(x) > 0, \textit{ No solution found.}$$
5. *The first derivative of f determined by software has the following expression:*
   $$f'(x) = 3\cos^2 x \sin x + 3\cos x - 3\cos^3 x. \tag{2}$$
   Classically, the first derivative of f is determined as
   $$f'(x) = 3\cos^2 x \sin x + 3\sin^2 x \cos x = 3\cos x \sin x(\cos x + \sin x). \tag{3}$$

The last two different expressions of f cannot be compared by using software:

$3\cos^2 x \sin x + 3\cos x - 3\cos^3 x = 3\cos x \sin x(\cos x + \sin x)$ *is undecidable,*

or

$f'(x) = 3\cos x \sin x(\cos x + \sin x)$ *is undecidable.*

Therefore, these two expressions have to be compared by usual calculations. Starting from the expression $3\cos x \sin x(\cos x + \sin x)$ we get

$$\begin{aligned} 3\cos x \sin x(\cos x + \sin x) &= 3\cos^2 x \sin x + 3\cos x \sin^2 x \\ &= 3\cos^2 x \sin x + 3\cos x - 3\cos^3 x, \end{aligned}$$

i.e., the same expression obtained by the software.

Starting from the expression $3\cos^2 x \sin x + 3\cos x - 3\cos^3 x$, the students claimed that it was more difficult to make the following transformation (without doing the previous one)

$$\begin{aligned} 3\cos^2 x \sin x + 3\cos x - 3\cos^3 x &= 3\cos^2 x \sin x + 3\cos x \sin^2 x \\ &= 3\cos x \sin x\,(\cos x + \sin x)\,. \end{aligned}$$

The software determined the zeroes of the first derivatives as follows:

$f'(x) = 0$, *Solution is :*

$$\left\{x = \frac{1}{2}\pi\right\}, \quad \{x = 0\}, \quad \{x = \pi\}, \quad \left\{x = -\frac{1}{4}\pi\right\}, \quad \left\{x = \frac{3}{4}\pi\right\}. \tag{4}$$

From the expression (3) it follows that $f'(x) = 0$, for $x = \frac{3\pi}{2} = 4.7124$ also, but this point is not determined by software. Therefore, it is necessary to check out all obtained results. Considering the interval $[0, 2\pi]$, function f' has zero for $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$. Let us remark that classical solving of the equation of the form $3\cos^2 x \sin x + 3\cos x - 3\cos^3 x = 0$, (where the form of the first derivative is obtained by using the software), is more complicated than solving the equation $3\cos x \sin x\,(\cos x + \sin x) = 0$.

By using software we can, also, try to solve the equation

$$3\cos^2 x \sin x + 3\cos x - 3\cos^3 x = 0, \quad \textit{Solution is :}$$
$$\left\{x = \frac{1}{2}\pi\right\}, \{x = 0\}, \{x = \pi\}, \quad \left\{x = -\frac{1}{4}\pi\right\}, \left\{x = \frac{3}{4}\pi\right\},$$

and we obtain the same results as (4).

In this case the students have to do even more calculation and trigonometry transformations than in a classical way, but they claim that, now this is much more interesting, for them, because by using the graph of f they found that it has minimum at $x = \frac{3\pi}{2}$, which is omitted in software results.

On Figure 4, the function f is drawn with the solid line and the first derivative is given by dashes. We can see on the graph that the extremes of f are attained

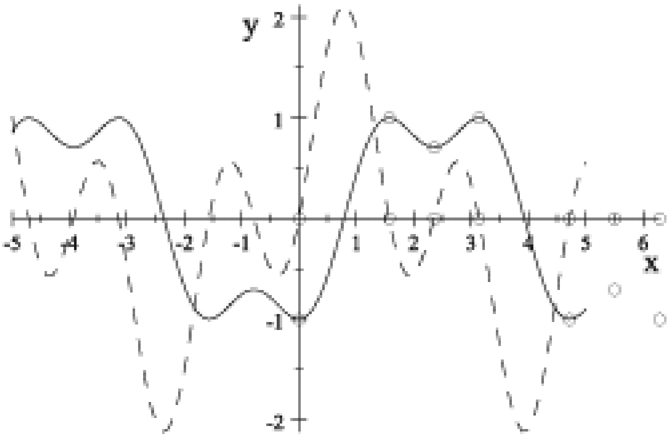

Fig. 4

at the points where f' has zeroes. The sign of the first derivative is difficult to be determined classically, *and the software cannot make that:*

$f'(x) > 0$, *No solution found.*

Looking at the graph, on Figure 4, it can be seen that:

- the function f (solid line) increases for $x \in (0, \frac{\pi}{2})$, $x \in (\frac{3\pi}{4}, \pi)$, $x \in (\frac{3\pi}{2}, \frac{7\pi}{4})$, where the first derivative (dashes) f' has positive values.
- the function f (solid line) decreases for $x \in (\frac{\pi}{2}, \frac{3\pi}{4})$, $x \in (\pi, \frac{3\pi}{2})$, $x \in (\frac{7\pi}{4}, 2\pi)$, where the first derivative (dashes) f' has negative values.

6. *The second derivative determined by software has the form*

$$f''(x) = 9\cos^2 x \sin x + 9\cos^3 x - 3\sin x - 6\cos x.$$

From

$$f''(\frac{1}{2}\pi) = -3, \qquad f''(\pi) = -3, \quad f''(\frac{7}{4}\pi) = -\frac{3}{2}\sqrt{2},$$
$$f''(\frac{3}{4}\pi) = \frac{3}{2}\sqrt{2}, \quad f''(0) = 3, \qquad f''(\frac{3}{2}\pi) = 3,$$

it follows that the function f has maximum at $x = \frac{1}{2}\pi$, $x = \pi$, $x = \frac{7}{4}\pi$, minimum at $x = 0$, $x = \frac{3}{4}\pi$, $x = \frac{3}{2}\pi$.

Using the software we get

$f''(x) = 0$, *Solution is :* $\left\{x = \frac{1}{4}\pi\right\}, \left\{x = -\frac{3}{4}\pi\right\}, \left\{x = \arctan\left(3\rho^3 - 3\rho, \rho\right)\right\}$ *where ρ is a root of* $9Z^4 - 9Z^2 + 1$.

In this case the software could not help us to determine all zeros of the second derivative, because from the graph it can be concluded that the function f have the saddle points. Solving equation

$$9\cos^2 x \sin x + 9\cos^3 x - 3\sin x - 6\cos x = 0,$$

is more complicated than solving the equation

$$3(\cos x - \sin x)\left(1 + \frac{3\sin 2x}{2}\right) = 0. \tag{5}$$

The last expression of the second derivative is obtained classically. The solutions of equation (5) are

$$\begin{aligned} x &= \frac{1}{4}\pi = 0.785\,40, & x &= \frac{5}{4}\pi = 3.\,927\,0,\\ x &= -\frac{1}{2}\arcsin\frac{2}{3} + \pi = 2.\,776\,7, & x &= -\frac{1}{2}\arcsin\frac{2}{3} + 2\pi = 5.\,918\,3,\\ x &= \frac{1}{2}\arcsin\frac{2}{3} = .\,364\,86, & x &= \frac{1}{2}\arcsin\frac{2}{3} + \pi = 3.\,506\,5. \end{aligned}$$

The last four values we get as solutions of equation

$$1 + \frac{3\sin 2x}{2} = 0.$$

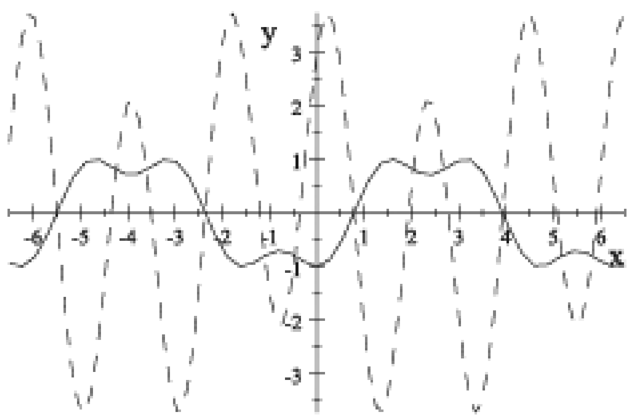

Fig. 5

On the Figure 5, the graph of the function f is drawn with solid line and the graph of its second derivative f'' is given by dashes.

From the graph it can be seen that the saddle points of f are attained for the zeroes of f'', and the dependence of concavity on the second derivative.

3. Function $h(x) = \sin x - \cos x$

It is interesting to compare function f with the function $h(x) = \sin x - \cos x$.

The graphs of f and h are drawn on Figures 6, 7, the graph of f is drawn with the solid line and the graph of function h is drawn with dashes.

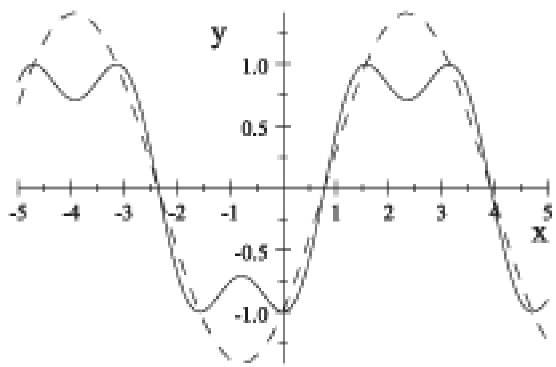

Fig. 6

1. Both functions are defined for all real values, being periodic with the main period 2π.
2. They are neither even nor odd functions.
3. They both have zeroes for $x = \frac{\pi}{4}$, and $x = \frac{5\pi}{4}$.

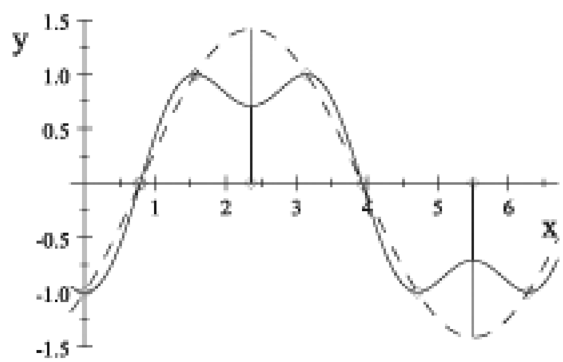

Fig. 7

- Besides the mentioned points these two functions have also common points for $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, $x = \frac{7\pi}{2}$, where either $\sin x = 0$, $\cos x = 1$, or $\cos x = 0$, $\sin x = 1$.
- They both have extremes at the points $x = \frac{3\pi}{4}$, and $x = \frac{7\pi}{4}$, but the function f has minimums whereas the function h has maximums at these points.

4. The function $g(x) = \sin x \sin 3x$

1. From $g(x) = \sin x \sin 3x$, and $g(x+\pi) = \sin x \sin 3x$ it follows that function g has the main period π.

On the Figure 8, the graph of function is drawn by default, and on Figure 9 it is drawn on the interval $[0,\pi]$.

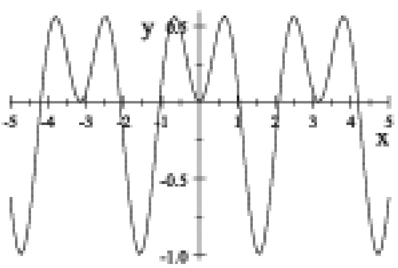

Fig. 8

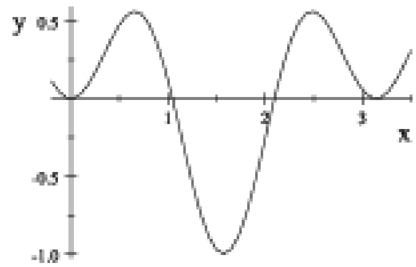

Fig. 9

2. The function g is the product of two odd functions and is therefore an even one.
3. The function g can be written as

$$g(x) = 4\sin^2 x \cos^2 x - \sin^2 x = \left(\sin^2 x\right)(2\cos x - 1)(2\cos x + 1) \tag{6}$$

The zeroes obtained by software are:

$g(x) = 0$, *Solution is :* $\{x = 0\}, \left\{x = \frac{1}{3}\pi\right\}, \left\{x = -\frac{1}{3}\pi\right\}, \left\{x = \frac{2}{3}\pi\right\}, \left\{x = -\frac{2}{3}\pi\right\}$.

It can be remarked that the software has treated cosine function as even one.

Since the function has the smallest period π, it is enough to consider the zeroes

$$x = 0 \qquad x = \frac{1}{3}\pi \qquad x = \frac{2}{3}\pi. \tag{7}$$

Classically, it is easy to solve equation $\sin x \sin 3x = 0$.

The same results, as in (6) are obtained from (7), manually, but the students had to check that.

4. From

$$\begin{aligned} g'(x) &= \cos x \sin 3x + 3\sin x \cos 3x = 6\cos^3 x \sin x - 10\cos x \sin^3 x \\ &= 2(\sin x \cos x)\left(3\cos^2 x - 5\sin^2 x\right) \end{aligned}$$

or

$$g'(x) = 16\sin x \cos^3 x - 10\cos x \sin x = 2(\cos x \sin x)\left(8\cos^2 x - 5\right)$$

(the same expression is obtained classically and *by using software*) we get

$g'(x) = 0$, *Solution is:* $\{x = 0\}$, $\left\{x = \frac{1}{2}\pi\right\}$, $\left\{x = -\frac{1}{2}\pi\right\}$,

$\left\{x = \arctan \frac{1}{10}\sqrt{6}\sqrt{10}\right\}$, $\left\{x = -\arctan \frac{1}{10}\sqrt{6}\sqrt{10}\right\}$,

$\left\{x = -\arctan \frac{1}{10}\sqrt{6}\sqrt{10} + \pi\right\}$, $\left\{x = \arctan \frac{1}{10}\sqrt{6}\sqrt{10} - \pi\right\}$.

Considering the interval $[0, \pi]$, we can write $g'(x) = 0$, for

$x = 0$, $\quad x = \frac{1}{2}\pi = 1.5708$, $x = \arctan \frac{1}{10}\sqrt{6}\sqrt{10} = .65906$,

$x = -\arctan \frac{1}{10}\sqrt{6}\sqrt{10} + \pi = 2.4825$.

Working classically, without computer, we have to solve the equation $8\cos^2 x - 5 = 0$, Solution is :

$\left\{x = \arccos \frac{1}{4}\sqrt{10}\right\}$, $\left\{x = \pi - \arccos \frac{1}{4}\sqrt{10}\right\}$.

The equation $8\cos^2 x - 5 = 0$, is equivalent to the equations

$$\cos^2 x = \frac{5}{8}, \text{ or } \quad \frac{1}{\cos^2 x} = \frac{8}{5}, \text{ or } \quad 1 + \tan^2 x = \frac{8}{5}, \text{ or } \quad \tan^2 x = \frac{3}{5}.$$

The solution of the last equation is: $\quad x = \pm \arctan \sqrt{\frac{3}{5}}$.

Furthermore, we can calculate

$\arctan \sqrt{\frac{3}{5}} = .65906 = \arccos \frac{1}{4}\sqrt{10} = .65906$,

$\pi - \arccos \frac{1}{4}\sqrt{10} = 2.4825$.

Such a combination of classical calculation and the calculation with the software contributes to the examining of trigonometry identities, as well.

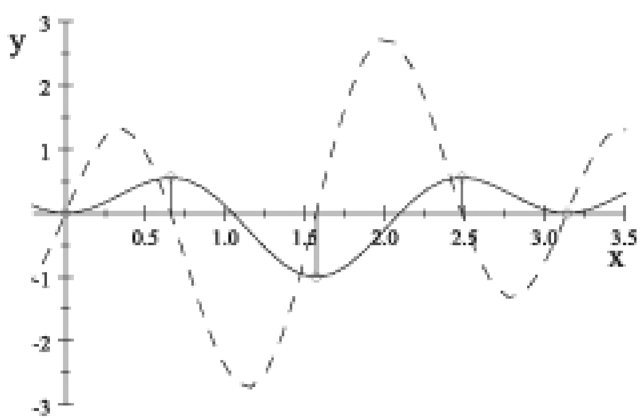

Fig. 10

On Figure 10 the graph of the function g is drawn by solid line and the graph of g' is drawn with dashes.

Looking at the graph, on Figure 10, it can be seen that:

- the function f (solid line) increases for

$$x \in \left(0, \arctan\sqrt{\frac{3}{5}}\right), \quad x \in \left(\frac{\pi}{2}, \pi - \arctan\sqrt{\frac{3}{5}}\right), \quad x \in \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right),$$

where the first derivative (dashes) f' has positive values;

- the function f (solid line) decreases for

$$x \in \left(\arctan\sqrt{\frac{3}{5}}.\frac{\pi}{2}\right), \quad x \in \left(\pi - \arctan\frac{1}{10}\sqrt{6}\sqrt{10}, \pi\right),$$

where the first derivative (dashes) f' has negative values.

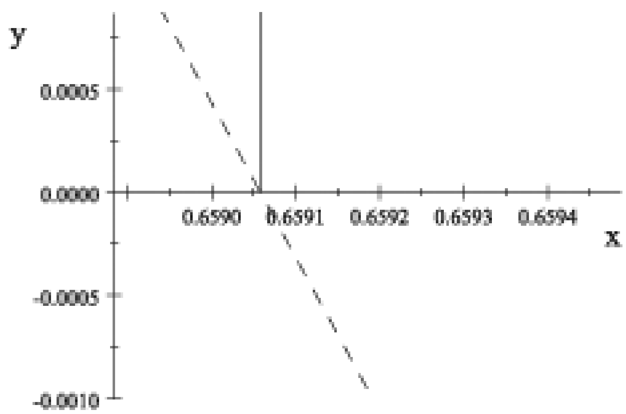

Fig. 11

On Figure 11 it is shown that the zeroes of the first derivative can be determined graphically, by enlarging the picture of the interval surrounding the zero of the first derivative.

Graphically, the first zero of the first derivative can be obtained as on the Figure 11, i.e., $x = 0.65905$.

5. Classically and *by using software* we get

$g''(x) = 6\cos x\cos 3x - 10\sin x\sin 3x$.

$g''(0) = 6, \quad g''(\frac{1}{2}\pi) = 10 \quad g''(\arctan\frac{1}{10}\sqrt{6}\sqrt{10}) = -\frac{15}{2}$

$g''(\arctan\frac{1}{10}\sqrt{6}\sqrt{10} - \pi) = -\frac{15}{2}$,

whereby we get that the considered function has:

minimums at $x = 0$, and $x = \frac{\pi}{2}$,

maximums at $x = \arctan\dfrac{1}{10}\sqrt{6}\sqrt{10}$, and $x = \arctan\frac{1}{10}\sqrt{6}\sqrt{10} - \pi$.

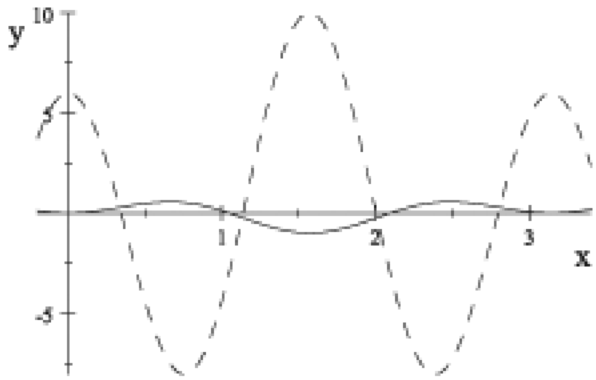

Fig. 12

The software determines:

$g''(x) = 6\cos x\cos 3x - 10\sin x\sin 3x = 6\cos^4 x - 48\cos^2 x\sin^2 x + 10\sin^4 x.$

In this example the students have problems with determining the zeroes of the second derivative. *The results obtained by software are*

$g''(x) = 0$, *Solution is:* $\left\{x = \arccos\frac{1}{8}\sqrt{\left(34+2\sqrt{129}\right)}\right\}$,

$\left\{x = \pi - \arccos\frac{1}{8}\sqrt{\left(34+2\sqrt{129}\right)}\right\}$, $\left\{x = \arccos\frac{1}{8}\sqrt{\left(34-2\sqrt{129}\right)}\right\}$,

$\left\{x = \pi - \arccos\frac{1}{8}\sqrt{\left(34-2\sqrt{129}\right)}\right\}$,

or numerically, we get

$x_1 = \arccos\frac{1}{8}\sqrt{\left(34+2\sqrt{129}\right)} = .344\,12$

$x_2 = \pi - \arccos\frac{1}{8}\sqrt{\left(34+2\sqrt{129}\right)} = 2.\,797\,5$

$x_3 = \arccos\frac{1}{8}\sqrt{\left(34-2\sqrt{129}\right)} = 1.\,137\,5$

$x_4 = \pi - \arccos\frac{1}{8}\sqrt{\left(34-2\sqrt{129}\right)} = 2.\,004\,1.$

Each value can be checked on the graph given on the Figure 12.

4. The conclusion

The conclusion is that the software is of great help for drawing and examining the graph of function, but the use of software needs careful control. The students learned that they have to be very careful with the results obtained by software and to check them exactly or approximately.

It is interesting to mention that the students enjoyed examining the functions interactively, combining results obtained by software with the usual calculation, even though they sometimes had much more to do. Namely, the numerical results obtained by software were supposed to be checked. The students claimed that checking results, especially detecting the "mistakes" obtained by computers is more interesting than just counting. In our opinion, the "mistakes" for the considered problems were pointed out, and they were removed by using the classical procedure.

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