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STUDENTS' UNDERSTANDING OF THE VARIABLE AS GENERAL NUMBER AND UNKNOWN: A CASE STUDY

Levent Akgün and M. Emin Özdemir

Abstract. This study aims at determining the misconceptions and learning difficulties that students experience regarding general number and unknown which are different aspects of the use of the concept of variable. Two sub-problems were examined for this purpose. The first one is the comments of students on general number and the second one is their comments on unknown. A qualitative method (case study design) was used in this study. The data presented in this paper were obtained through the analysis of the students' answers to the tests designed in accordance with the examined issues, their work on the variable-related questions in the exam documents, and the interviews with some of them. The sample of this research consisted of 158 eight grade students. The results show that the subjects of this study experience a set of difficulties and have certain misconceptions in understanding the concept of variable. This paper presents these learning problems and considers means to improve the matters.

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Key words and phrases: Different uses of variable, General number, Unknown.

Introduction and theoretical framework

Experience of teachers and several experimental studies show that students have difficulties in understanding general arithmetic algebra (Haspekian 2003). There are several conceptual obstacles in proceeding in algebra (Linchevski and Hercovics 1996, Stacey and MacGregory 1997, Thomas and Tall 1991). And one of the most important difficulties is the failure in understanding the concept of variable. This concept is rarely discussed in most of the classes where algebra courses are given. However this concept is the basis of everything that students learn. While most of the basic textbooks do not explain and even mention the concept of variable, Skemp (1986, p. 213) stressed that "A variable is in fact a key concept in algebra". Also, despite the importance of the concept, many mathematics curriculums continue discussing variables like simple terms (Kieran 1981). Research thus shows that students experience difficulties regarding the concept of variable (Schoenfeld and Arcavi 1988, Bills 2001, Cooper et al. 1997). If we want students go beyond basic arithmetic operations, they should fully understand the concept of variable before starting with other algebraic concepts. One of the reasons that makes the concept of variable difficult is the frequent use of "letters" and "verbal symbols". "Letters" and "verbal symbols" used in algebra are complex and multiple representations of this concept (Schoenfeld and Arcavi 1988) and, at least due to such representations, it is a difficult task to express it. Skemp (1986) describes variable as "an unspecified element of a given set" (p. 214).

Several researchers stated that the concept of variable is one of the most essential concepts in taught mathematics; beginning with primary school and continuing throughout high school (see Sasman et. al. 1997). Philipp (1992) mentions Rajaratnam who argued in the late fifties that this concept is so important that its discovery was a milestone in the history of mathematics. Regarding the same issue, Percy Nunn argued in 1919 that discovery of variables is probably the most important event in the history of humanity and the sovereignty of their use will remain as one of the most important successes of the history of humanity. About ten years latter, William Betz reported that symbolization in algebra is its reputation and honor despite that this symbolization is also its curse (see Sasman et al. 1997).

The concept of variable is closely related to the development of the concept of function. In fact, as Kline underlined, Leibniz was the first person who touched on the terms function and variable (see Philipp 1992). The close relationship between the concept of variable and function continued to exist in the school practice during the first half of the 20th century. Also, this situation can be easily understood from the following definition: "related numbers like x and y that vary together are called variables//. Philipp (1992) also gave Osborne's definitions of variable: "a quantity that can take infinite numerical values". Many textbooks distinguished quantities that represent a unique value (constants) from quantities that can stand for several values (variables). For example, in the canonical equation of the circle, $x^2+y^2=a^2$, x and y are variables and a is constant (see Philipp 1992).

The reformation in mathematics in the late 1950s and early 1960s brought about a significant change in the definition of variable so that this change is still effective today. As opposed to the unified concepts in the mathematics curriculum, the concept of variable has been taught in its general form since the beginning and all letter symbols have been called variable (Kieran 1989). Since then, the concept of variable has not been associated with the concept of function but rather with the concept of set. According to Tonnessen's study that analyzed mathematical textbooks published between the early 1950s and late 1980s, almost all textbooks made a definition of variable directly or indirectly and their variable was a symbol that represents any of the members of a set that has at least two elements (see Philipp 1992). In other words, almost all uses of letter symbols were variables. By taking such a position, letter symbol in the relation $x+3=7$, x is a variable because x represents any of the elements of a set whose domain is not specified but is usually assumed (inferred from the underlying context). In this case can be a real, rational, integer or natural number. Let us recall that x is a variable providing that its domain has at least two elements. Only examples of letters that are not variables are symbols representing special numbers, such as e – the base of natural logarithm, c – the velocity of light and π – the ratio of the circumference of a circle and its diameter (see Philipp 1992).

Today, many researchers who are interested in mathematics education have not accepted a common definition of the concept of variable. They rather suggest different definitions related to this concept. (One of them, taken from Philipp (1992, p. 26), says that variable is "a symbol which can represent any of the elements a specified set"). As a result, curriculum designers usually face different opinions about the definition of the concept of variable and their approach to this concept may cause many constraints in learning and teaching algebra. We should thus fully understand the definitions used for the concept of variable. Understanding the concept of variable has a vital importance in constituting a strong basis for other algebraic concepts related to it (Balyta 1999). According to Graham and Thomas (2000), this understanding is the basis of all advanced studies. It is thus profoundly important for all students to gain self-confidence in using variables. Since variable is a concept that requires a rich concept-image (Tall and Vinner 1981), students are not expected to understand that concept easily.

Methodology

This study aimed at determining the misconceptions and learning difficulties that students experience regarding general number and unknown, which are among different uses of the concept of variable. The following issues were examined for this purpose.

- Comments of students on general number, and
- Comments of students on the concept of unknown.

In order to examine these issues in detail, a case study approach was applied (Yıldırım and Şimşek 2000; Mcmillan and Schumacher 2001; Çepni, 2001). The data presented in this paper were obtained through the analysis of the students' answers to the tests designed in accordance with the examined issues, their work on the variable-related questions in the exam documents, and the interviews with some of them (found appropriate for our study). These data were used to determine the students' level of knowledge related to the concept of variable and its different uses as well as their misconceptions and learning difficulties.

The sample of this research consisted of 158 eight grade students, who were selected from a population by applying set sampling- a probability-based sampling method examined in Yıldırım and Şimşek (2000), for example.

In order to strengthen both reliability and validity of the applied tests, we administered them to other eight grade students (outside our sample) and revised or amended questions that found challenging. The subjects' answers to the questions in the revised tests were categorized as "correct", "incorrect" and "unanswered" and occasionally also as "semi-correct". We will present the frequencies of these categories for selected questions and discuss possible reasons for the presented findings.

Findings

Variable as a general number

As it is already mentioned, the concept of variable has been defined in association with the concept of set since the 1950s. In one question, the subjects were asked to state all values that x can take in the relation "$x+2=2+x$". Most of the students perceived this relation as an equation and tried to solve it. Because of that 39 students claimed that x stood for one number. The interviews showed that the existence of a statement about x caused the subjects to perceive the given relation as an equation. Both the interviews and the analyzed exam documents evidenced that, as soon as the subjects saw $x, y, \ldots$ in the given relations, they tried to solve the corresponding equations. Only 25 students (15% of the sample) perceived variable x in that relation as a general number and 19 of them (just 12% of the sample) gave the correct answer (x can be any real number). The conducted interviews and the considered exam documents evidenced that these students hold the idea of variable as general number. As regards partially correct (or semi-correct) answers, 30 students said that x stood for positive integers and natural numbers, and 12 that x stood for integers. It was surprising to observe that although the subjects knew real numbers as the widest numerical set, they had in their minds integers or natural numbers as such a set while answering this question. Also, twenty students said that x stood for infinite values, but the interviews showed that these students perceived the concept of general number in very different ways: some of them thought of x as any natural number, some as any integer, and some as one number or any of many numbers. Note that eleven students left this question unanswered.

Another question concerning the use of variable as general number was "When does the relation $x + y = z + y$ hold true?" Thirty five students answered to this question wrongly: ten students chose "always", whereas 25 replied "never". Only 31 students (20% of the sample) answered "sometimes" with correct explanation "for each value of y when $x = z$". Although most students (78 out of 158) also answered "sometimes", they did not give any explanation when it happens. Note that two students tried to verify the relation by giving concrete values to the three variables (e.g. $2+1 = 3+1$), whereas twelve students did not answer that question.

It is clear that our students experienced difficulties in understanding and using variable as general number. Similar findings have been obtained in other studies. Ursini (1990) reported that students experience major difficulties and lack of confidence when they face different kinds of variable (one of them is variable as general number). Redden (1996) concluded that not all children have the ability to generalize, which is, for example, manifested when a symbol (variable) can be used instead of any number.

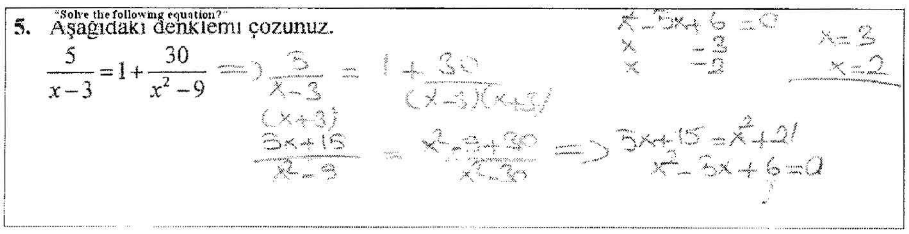
Variable as unknown

Unknowns are variables that represent specific numbers. To understand such a use of variable, students should be able to:

- recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of that situation;
- interpret symbols that appear in equations as the representatives of specific numbers;
- substitute variables in equations with numbers (one or several) to produce relations that are true numerical statements;

- determine unknown quantities that appear in equations by performing suitable algebraic and arithmetic operations; and
- symbolize unknown quantities identified in problem situations and use such symbols to pose equations (see Ursini and Trigueros 2001).

One of the questions examining the understanding of variable as unknown was "Solve equation $\frac{5}{x-3} = 1 + \frac{30}{x^2-9}$". Only ten subjects could answer this question correctly. An example of an almost correct answer is given below (the answer is not fully correct because $x = 3$ must be discarded; otherwise the validity check will result in the division by zero).



Although 50 subjects did not solve this equation, the work of the other subjects (98; 62% of the sample), who tried to solve this equation but did not succeed in doing this, enabled us to recognize main learning problems: These were:

- finding just one numerical value for caused by a wrong conceptualization of the degree of the underlying simpler equation (first instead of second degree);
- inability to transform the given equation into a more familiar form ($ax^2 + bx + c = 0$) caused by wrong algebraic transformations that were found to have roots in poorly-attained arithmetic operations with fractions; and
- determining the value of the unknown by trying out some concrete numbers.

Most frequent errors regarding the applied algebraic transformations were found in summing or subtracting expressions with the unknown such as $5x^2 - 3x = 2x^2$. These students did forget an old rule "don't sum apples and pears" that was to be applied in an algebraic context. In general, the students' answers and the conducted interviews evidenced very weak procedural and conceptual knowledge concerning the operations with unknowns.

Misconceptions and incorrect procedures have been treated as "bugs" in conceptual and procedural knowledge, respectively. A model of algebraic competence suggest that procedural bugs occur when the learner goes beyond a known rule not being able to evaluate the appropriateness of a particular extrapolation (Matz, 1980). To have more didactically-valued findings, research should start focusing on finding out known objects (concepts or procedures) and problematic situations that are likely to be the origins of particular bugs.

Closing remarks

The concept of variable is a multifaceted construct because it can, as exemplified here, be seen as general number or unknown.

Our subjects had difficulties in distinguishing the concepts of identity from that of equation—they considered variables in identities as unknowns and tried to "solve" these identities as they were equations—which prevented them to understand variable as general number. The results also showed that the subjects were not able to relate a variable to different sets of numbers (natural numbers, integers, etc.). To improve the matters, the concept of set should be better taught and learned. Also, simple programming activities can be utilized due to their promising values concerning the understanding of variables (see Ortiz & MacGregor, 1991).

Our subjects had also misconceptions and learning difficulties in understanding and making use of variable as unknown; particularly in carrying out algebraic transformations where they freely "found" sums and differences of "apples and pears". It is clear that our subjects had not attained the meaning of basic expressions such as $2x$, $3x^2$ etc., which naturally resulted in absurdities such as $x^2 + 2x = 2x^3$. To improve the matters, regular mathematics education must establish the meaning of basic algebraic expressions, making use of main links between arithmetic and algebra (see Marjanovic & Kadijevich, 2001). To improve students' algebraic competences (there are usually weak in general), a carefully-designed computer assisted learning of algebraic transformation should be applied (see Yerushalmy, 1991).

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Department of Mathematics Education, Faculty of Kazım Karabekir Education, Atatürk University, 25240-Erzurum/Turkey

E-mail: levakgun@gmail.com, emos@atauni.edu.tr